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CRC Preprint 2025/3, January 2025

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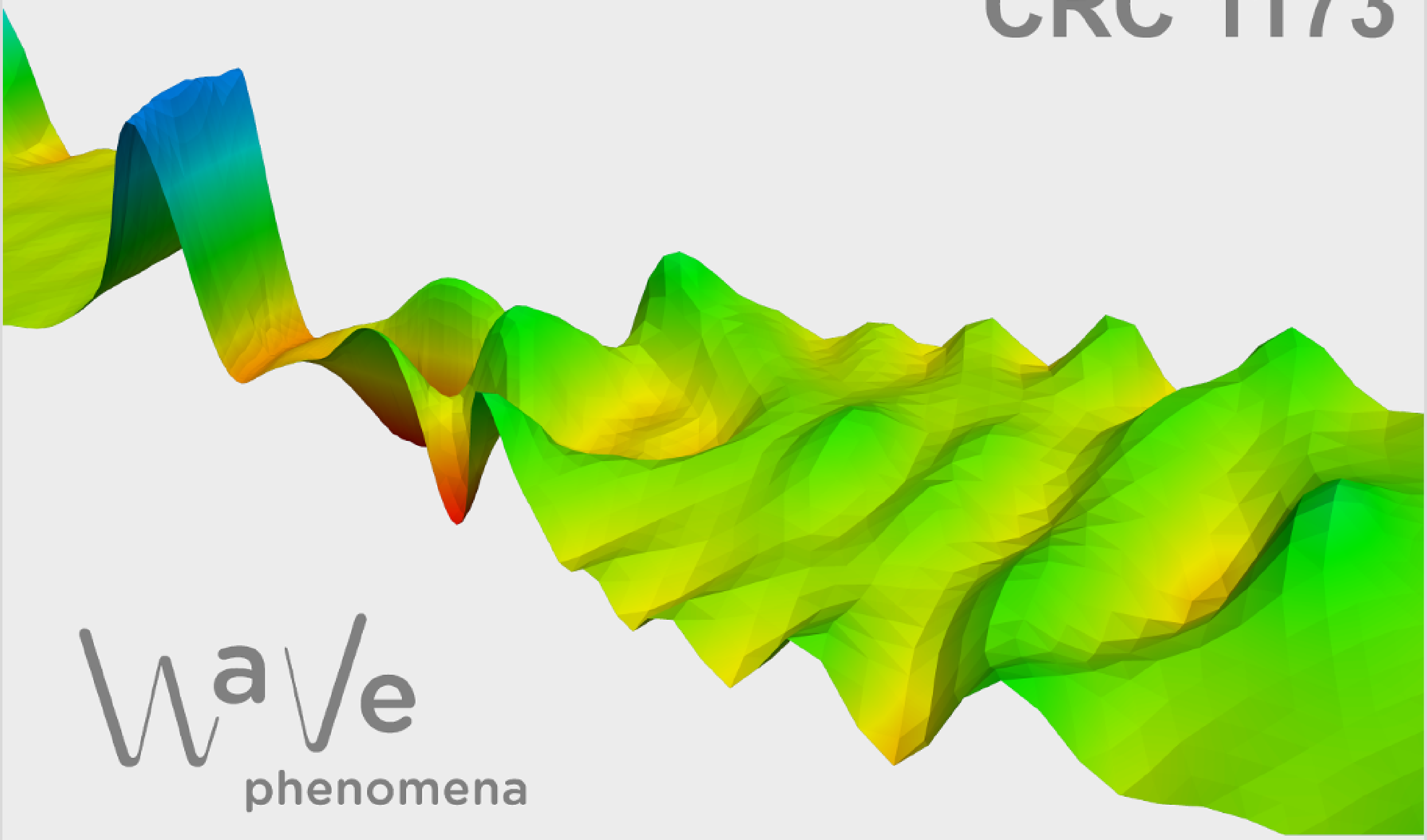
CRC 1173



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ISSN 2365-662X

FINITE ELEMENT DISCRETIZATION OF NONLINEAR MODELS OF ULTRASOUND HEATING

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Abstract. Heating generated by high-intensity focused ultrasound waves is central to many emerging medical applications, including non-invasive cancer therapy and targeted drug delivery. In this study, we aim to gain a fundamental understanding of numerical simulations in this context by analyzing conforming finite element approximations of the underlying nonlinear models that describe ultrasound-heat interactions. These models are based on a coupling of a nonlinear Westervelt–Kuznetsov acoustic wave equation to the heat equation with a pressure-dependent source term. A particular challenging feature of the system is that the acoustic medium parameters may depend on the temperature. The core of our new arguments in the *a prior* error analysis lies in devising energy estimates for the coupled semi-discrete system that can accommodate the nonlinearities present in the model. To derive them, we exploit the parabolic nature of the system thanks to the strong damping present in the acoustic component. Theoretically obtained optimal convergence rates in the energy norm are confirmed by the numerical experiments. In addition, we conduct a further numerical study of the problem, where we simulate the propagation of acoustic waves in liver tissue for an initially excited profile and under high-frequency sources.

1. Introduction

High-intensity focused ultrasound (HIFU) waves are known to act as a source of heat within the body. This heating phenomenon is at the core of many developing medical applications, including non-invasive ablation of cancer and targeted drug delivery; see, e.g., [2, 8] for details. Rigorous mathematical research into the underlying (inherently nonlinear) models of wave-heat interactions has been initiated relatively recently with the contributions of, e.g., [1, 1, 2], which have investigated local and global well-posedness of the exact models. To the best of our knowledge, rigorous numerical understanding in this context is currently missing in the literature. In this work, we investigate conforming finite element approximations of the underlying models and develop a theoretical framework for their *a priori* error analysis. Ultrasound-heat interactions present in HIFU-induced heating can be captured using a coupled system based on a damped nonlinear acoustic equation and the heat equation as follows:

$$\begin{cases} u_{tt} - c^2(\theta)\Delta u - \beta(\theta)\Delta u_t + \mathcal{N}(u, u_t, u_{tt}, \nabla u, \nabla u_t, \theta) = f, & \text{in } \Omega \times (0,T), \\ \theta_t - \kappa\Delta\theta + \nu\theta = \mathcal{Q}(u, u_t, \theta), & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

with damping coefficients κ, $\nu > 0$. Heating occurs due to the acoustic energy that is absorbed by the tissue, which is here modeled by having a pressure-dependent source term $\mathcal{Q}$ in the heat equation. The acoustic medium parameters are known to depend on the temperature, resulting in the so-called *thermal lensing* effect, where the focal region of the ultrasound waves may shift with

2020 *Mathematics Subject Classification.* 35L05, 35L72, 34A34.

Key words and phrases. Westervelt's equation, Kuznetsov's equation, wave-heat coupling, finite element approximation, a priori analysis.

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changes in the temperature. In particular, this temperature dependency is seen in the speed of sound c and sound diffusivity β; the latter is computed using the relation

$$\beta = 2\frac{\tilde{\alpha}c^3}{\omega^2},$$

where $\tilde{\alpha} = \tilde{\alpha}(\theta)$ denotes the acoustic amplitude absorption coefficient and ω the angular frequency; see [5].

The acoustic wave equation in (1.1) generalizes the classical Westervelt and Kuznetsov equations in nonlinear acoustics. In the case of the Westervelt equation, u in (1.1) represents the acoustic pressure p, whereas in the Kuznetsov equation u represents the acoustic velocity potential ψ. The two quantities can be related using $p = \rho\psi_t$, where ρ is the medium density. Concerning the nonlinearities, these two classical equations are recovered with the following choices:

$$\mathcal{N} = \begin{cases} k_{\mathrm{W}}(\theta)\left(u^2\right)_{tt} = 2k_{\mathrm{W}}(\theta)\left(uu_{tt} + u_t^2\right) & \text{Westervelt's equation,} \\ k_{\mathrm{K}}(\theta)\left(u_t^2\right)_t + (|\nabla u|^2)_t = 2k_{\mathrm{K}}(\theta)u_t u_{tt} + 2\nabla u \cdot \nabla u_t & \text{Kuznetsov's equation,} \end{cases}$$

where the temperature-dependent nonlinearity coefficients are given by

$$k_{\mathrm{W}} = \frac{1}{\rho c^2}(1 + \tfrac{B}{2A}), \qquad k_{\mathrm{K}} = \frac{1}{c^2}\frac{B}{2A}. \tag{1.2}$$

In (1.2), $\frac{B}{A}$ represents the acoustic nonlinearity parameter of the medium. Note that this parameter is also known to depend on the temperature; see, for example, [2, Fig. 7].

1.1. **Mathematical generalization of the model.** To encompass both nonlinearity cases we assume in the analysis that the functional $\mathcal{N}$ is given by

$$\mathcal{N}(u, u_t, u_{tt}, \nabla u, \nabla u_t, \theta) = k_{\mathrm{W}}(\theta)\left(u^2\right)_{tt} + k_{\mathrm{K}}(\theta)\left(u_t^2\right)_t + \ell(|\nabla u|^2)_t, \quad \ell \in \mathbb{R}. \tag{1.3}$$

Concerning the nonlinearity coefficients in (1.3), we assume that

$$k_{\mathrm{W}},\, k_{\mathrm{K}} \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}).$$

Assumptions on the medium parameters. Let $q = c^2$. Regarding the temperature-dependent speed of sound and sound diffusivity, we assume that $q \in P_m(\mathbb{R})$ and $b \in P_n(\mathbb{R})$ are polynomials over $\mathbb{R}$ of maximal degree $m \in \mathbb{N}$ and $n \in \mathbb{N}$, respectively, and such that

$$q_0 := q(0) > 0, \quad \beta_0 := \beta(0) > 0.$$

These positivity assumptions correspond to the usual assumptions of positivity of the speed of sound and sound diffusivity at constant temperatures. We emphasize that the condition $\beta_0 > 0$ is particularly important as the presence of strong damping in the acoustic component (that is, having $-\beta_0\Delta u_t$) will allow us to employ parabolic estimates in the numerical analysis.

In practice, the speed of sound and the acoustic attenuation coefficient are indeed typically determined via a least-squares fit from data assuming polynomial dependence on the temperature; see, e.g., [1, 3, 5].

Assumptions on the absorbed energy. In the literature, different forms of the functional $\mathcal{Q}$ in (1.1) are employed; see, e.g., [1, 1, 5], and the references contained therein. We assume here that $\mathcal{Q}$ has the following form:

$$\mathcal{Q}(u, u_t, \theta) = \alpha(\theta)(\zeta_1 u^2 + \zeta_2 u_t^2) \quad \text{with } \alpha \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}), \quad \zeta_1, \zeta_2 \in \mathbb{R}.$$

This allows us to cover, for example, the plane wave approximation for the volume rate of heat deposition (see [1, eq. (10.2.11)]) given by $\mathcal{Q} = \frac{\tilde{\alpha}}{\rho c}p^2$ in both Westervelt and Kuznetsov regimes, where the acoustic pressure is $p = u$ and $p = \rho u_t$, respectively. Another expression for the absorbed energy found in the literature (see, e.g., [1]) is $\mathcal{Q} = \frac{2\beta}{\rho c^4}p_t^2$, which is covered here in the Westervelt regime, where $p = u$.

Remark 1.1. *Note that according to the available well-posedness results for the Westervelt–heat systems, the non-degeneracy condition*

$$q(\theta) \geq \underline{q} > 0$$

is expected to hold for sufficiently smooth and small pressure-temperature data. Thus, in the case $\alpha \sim \beta/q^2$*, the assumed regularity of* α *follows by the assumed properties of* q *and* β*.*

Notation. We use $x \lesssim y$ below to denote $x \leq Cy$, where $C > 0$ does not depend on the spatial discretization parameter h. By $(\cdot,\cdot)_{L^2}$ we denote the scalar product on $L^2(\Omega)$. We often omit the temporal domain $(0,T)$ when denoting the norms in Bochner spaces; for example, $\|\cdot\|_{L^p(L^q(\Omega))}$ denotes the norm on $L^p(0,T;L^q(\Omega))$. We use the subscript t to emphasize that the temporal domain is $(0,t)$ for some $t \in (0,T)$; for example, $\|\cdot\|_{L^p_t(L^q(\Omega))}$ denotes the norm on $L^p(0,t;L^q(\Omega))$ for $t \in (0,T)$.

1.2. **Assumptions on the exact solution.** Let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be an open and bounded set. Considering data, we assume homogeneous Dirichlet boundary conditions for the pressure and temperature and sufficiently regular initial pressure and temperature data. That is, we consider the approximation of the following initial-boundary value problem:

$$\begin{cases} u_{tt} - q(\theta)\Delta u - \beta(\theta)\Delta u_t + \mathcal{N}(u, u_t, u_{tt}, \nabla u, \nabla u_t, \theta) = f & \text{in } \Omega \times (0,T), \\ \theta_t - \kappa\Delta\theta + \nu\theta = \mathcal{Q}(u, u_t, \theta) & \text{in } \Omega \times (0,T), \\ u_{|\partial\Omega} = \theta_{|\partial\Omega} = 0, \\ (u, u_t)_{|t=0} = (u_0, u_1), \quad \theta_{|t=0} = \theta_0, \end{cases} \tag{1.4}$$

with $\mathcal{N}$ as in (1.3). Given $\eta \geq 1$ (which will denote the polynomial degree of the finite element basis functions on an element), we assume that there exists a unique solution of the problem such that

$$(u, \theta) \in \mathcal{X}_u \times \mathcal{X}_\theta,$$

with

$$\|u\|_{\mathcal{X}_u} + \|\theta\|_{\mathcal{X}_\theta} \leq C$$

for some $C > 0$, where the two spaces are defined as follows:

$$\begin{aligned} \mathcal{X}_u = \big\{u :\, & u \in L^\infty(0,T; H^{\eta+1}(\Omega) \cap W^{1,\infty}(\Omega) \cap H_0^1(\Omega)), \\ & u_t \in L^\infty(0,T; H^{\eta+1}(\Omega) \cap W^{1,\infty}(\Omega) \cap H_0^1(\Omega)) \\ & u_{tt} \in L^2(0,T; H^{\eta+1}(\Omega))\big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{X}_\theta = \big\{\theta : \theta \in\, & L^\infty(0,T; H^{\eta+1}(\Omega) \cap W^{1,\infty}(\Omega) \cap H_0^1(\Omega)) \cap L^2(0,T; W^{\eta+1,d+\delta}(\Omega) \cap W^{\eta+1,\infty}(\Omega)), \\ & \theta_t \in L^\infty(0,T; H^{\eta+1}(\Omega)) \cap W^{1,\infty}(\Omega)\big\} \end{aligned}$$

with $d + \delta \in [2,6]$. For the upcoming analysis, it is worth noting that

$$\mathcal{X}_u,\, \mathcal{X}_\theta \hookrightarrow L^\infty(0,T; L^\infty(\Omega)).$$

Furthermore, our main theoretical result (see Theorem 1.2) assumes that there exists a sufficiently small $r > 0$, such that

$$\|(\beta - \beta_0)(\theta)\|_{L^\infty(L^\infty(\Omega))} + \|k_{\mathrm{W}}(\theta)u\|_{C(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta)u_t\|_{C(L^\infty(\Omega))} + \|\alpha(\theta)u_t\|_{C(L^\infty(\Omega))} \leq r. \tag{1.5}$$

The small-data well-posedness analysis of (1.4) in the Westervelt case (and somewhat simplified function $\mathcal{Q}$) based on energy arguments can be found in [1, 1], under the assumption that the function q does not degenerate and that $\beta = \mathit{const.} > 0$. In [2], the concept of maximal L^p-L^q regularity has been utilized to show local and global well-posedness of the non-isothermal Westervelt equation. The small-data local and global well-posedness of the Kuznetsov equation with constant medium parameters can be found in [1]. Although wave-heat system (1.4) in its full generality assumed here does not appear to have been studied rigorously in the literature in terms of well-posedness, we expect that the general framework of [2] can be utilized for this purpose. The smallness assumption in (1.5) can then be enforced (via continuous dependence on data) by the smallness of initial data (u_0, u_1, θ_0).

1.3. **Main result.** We next present the main theoretical result of this work. We employ Lagrange finite elements here on a quasi-uniform triangulation $\mathcal{T}_h$ and introduce the finite element space incorporating the homogeneous boundary conditions

$$V_h \coloneqq \Big\{ \phi_h \in C_0(\Omega) \mid \phi_h|_K \in \mathcal{P}_\eta(K) \text{ for all } K \in \mathcal{T}_h \Big\} \tag{1.6}$$

of piecewise polynomials of degree $\eta \geq 1$, which is used both for the pressure and temperature. We introduce the Ritz projection defined for $\varphi \in H_0^1(\Omega)$ via

$$(\nabla \varphi, \nabla \phi_h)_{L^2} = (\nabla \mathrm{R}_h \varphi, \nabla \phi_h)_{L^2}$$

for all $\phi_h \in V_h$. Further, we rely on the nodal interpolation operator $\mathrm{I}_h \colon C(\Omega) \to V_h$, and define the discrete Laplacian operator $\Delta_h \colon V_h \to V_h$ for $\psi_h, \phi_h \in V_h$ via the relation

$$(\Delta_h \psi_h, \phi_h)_{L^2} = -(\nabla \psi_h, \nabla \phi_h)_{L^2}.$$

Further, we introduce the bilinear functional $a(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$ as follows:

$$a(\psi_h, \phi_h) = (\nabla \psi_h, \nabla \phi_h)_{L^2}.$$

With these preparations, we consider the semi-discrete acoustic problem:

$$\begin{aligned} &(\partial_t^2 u_h, \phi_h)_{L^2} + a(u_h, q(\theta_h)\phi_h) + a(\partial_t u_h, \beta(\theta_h)\phi_h) \\ &\qquad + (\mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h), \phi_h)_{L^2} = (f_h, \phi_h)_{L^2} \end{aligned} \tag{1.7a}$$

for all $\phi_h \in V_h$, $t \in [0,T]$, with

$$(u_h, \partial_t u_h)_{|t=0} = (u_{0h}, u_{1h}). \tag{1.7b}$$

Note that

$$\begin{aligned} a(u_h, q(\theta_h)\phi_h) &= a(u_h, \mathrm{R}_h[q(\theta_h)\phi_h]) = -(\Delta_h u_h, \mathrm{R}_h[q(\theta_h)\phi_h])_{L^2}, \\ a(\partial_t u_h, \tilde{\beta}(\theta_h)\phi_h) &= a(\partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h]) = -(\Delta_h \partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2}. \end{aligned}$$

The semi-discrete heat equation is given by

$$(\partial_t \theta_h, \phi_h)_{L^2} + \kappa a(\theta_h, \phi_h) + \nu(\theta_h, \phi_h)_{L^2} = (\mathcal{Q}(u_h, \partial_t u_h, \theta_h), \phi_h)_{L^2} \tag{1.8a}$$

for all $\phi_h \in V_h$, $t \in [0,T]$, with

$$\theta_{h|t=0} = \theta_{0h}. \tag{1.8b}$$

Our main theoretical result establishes *a priori* error bounds for (u_h, θ_h) in the energy norm.

Theorem 1.2 (*A priori* error estimate)**.** *Let the assumptions made on the temperature-dependent functions in Section 1.1 and on the exact solution (u,θ) of (1.4) in Section 1.2 hold with $\eta \geq 1$. Assume that f, $f_h \in L^2(0,T;L^2(\Omega))$ are such that*

$$\|f - f_h\|_{L^2(L^2(\Omega))} \leq Ch^\eta$$

and that the approximate initial data are chosen as the Ritz projections of the exact ones; that is,

$$(u_h(0), \partial_t u_h(0)) = (\mathrm{R}_h u_0, \mathrm{R}_h u_1), \quad \theta_h(0) = \mathrm{R}_h \theta_0.$$

Then, there exist $h_0 > 0$ and $r > 0$, independent of h, such that for all $h \leq h_0$ and

$$\|(\beta - \beta_0)(\theta)\|_{L^\infty(L^\infty(\Omega))} + \|k_{\mathrm{W}}(\theta)u\|_{C(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta)u_t\|_{C(L^\infty(\Omega))} + \|\alpha(\theta)u_t\|_{C(L^\infty(\Omega))} \leq r, \tag{1.9}$$

problem (1.7), (1.8) *has a unique solution* $(u_h, \theta_h) \in H^2(0,T;V_h) \times H^1(0,T;V_h)$, *which satisfies the following error bound:*

$$\begin{aligned}\|\partial_t^2(u-u_h)\|_{L^2_t(L^2(\Omega))} + \|\nabla\partial_t(u-u_h)\|_{L^\infty_t(L^2(\Omega))} + \|\nabla(u-u_h)\|_{L^\infty_t(L^6(\Omega))} \\ + \|\partial_t(\theta-\theta_h)\|_{L^\infty_t(L^2(\Omega))} + \|\nabla(\theta-\theta_h)\|_{L^\infty_t(L^6(\Omega))} \leq C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta})h^\eta.\end{aligned}$$

Discussion of the main result. Theorem 1.2 establishes sufficient conditions for the optimal order of convergence of (u_h, θ_h) in the energy norm. Let us discuss some of the assumptions. The need for using the Ritz projection of the initial values comes from bounds involving Δ_h applied to the initial error; see Sections 4 and 5. A different choice, say, for example, the nodal interpolation, would lead to an order reduction in the error analysis.

The assumption that the exact temperature should satisfy $\theta \in L^2(0,T;W^{\eta+1,d+\delta}(\Omega))$ comes from the need to estimate the error in the temperature dependent coefficients β and q. In particular, we need to employ the following estimate for $w \in \{q, \tilde{\beta}\}$ (see (5.3) below):

$$\begin{aligned}\|w(\theta) - w(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))} &\lesssim \|\theta - \theta_h\|_{L^2_t(W^{1,d+\delta}(\Omega))} \\ &\lesssim \|\theta_h - \mathrm{R}_h\theta\|_{L^2_t(W^{1,d+\delta}(\Omega))} + \|\theta - \mathrm{R}_h\theta\|_{L^2_t(W^{1,d+\delta}(\Omega))},\end{aligned}$$

and then further use the bound $\|\theta - \mathrm{R}_h\theta\|_{L^2(W^{1,d+\delta}(\Omega))} \lesssim h^\eta \|\theta\|_{L^2(W^{\eta,d+\delta}(\Omega))}$ (see Section 2.4 for the approximation properties of the Ritz projection).

Similarly, the assumption that $\theta \in L^2(0,T;W^{\eta+1,\infty}(\Omega))$ comes from needing to estimate the error in the temperature-dependent coefficients k_{W} and k_{K}. In particular, we will employ the following bound (see (5.4) below):

$$\begin{aligned}&\|k_{\mathrm{W}}(\theta) - k_{\mathrm{W}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta) - k_{\mathrm{K}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))} \\ \lesssim\, &\|\theta - \theta_h\|_{L^2_t(L^\infty(\Omega))} \\ \lesssim\, &\|\theta_h - \mathrm{R}_h\theta\|_{L^2_t(L^\infty(\Omega))} + \|\theta - \mathrm{R}_h\theta\|_{L^2_t(L^\infty(\Omega))}\end{aligned}$$

and then further need to rely on the fact that $\|\theta - \mathrm{R}_h\theta\|_{L^2(L^\infty(\Omega))} \lesssim h^\eta\|\theta\|_{L^2(W^{\eta+1,\infty}(\Omega))}$. Even though this regularity assumption could be improved, we still need it for a technical estimate within the proof of Lemma 4.2.

Finally, let us note that there is a large literature available on the discretization of nonlinear wave equations originating from the seminal work [1]. However, they do not consider the coupled wave-heat case, and we thus refrain from a further discussion.

1.4. **Organization of the rest of the paper.** The rest of the paper is organized as follows. In Section 2, we provide background results on parabolic estimates which are used in the well-posedness and error analysis of the semi-discrete problem, as well as certain useful properties of the Ritz projection and known embedding and inverse estimates. In Section 3, we prove that the semi-discrete problem has an accurate solution, however, on a possibly h-dependent time interval. Toward prolonging the existence of this solution to $[0,T]$, we then focus on deriving uniform energy estimates for the wave and heat subproblems in Sections 4 and 5, respectively. These are combined in Section 6 to prove of the main theoretical result of this work stated in Theorem 1.2. Finally, in Section 7 we validate the theoretical convergence rate through numerical experiments and provide additional numerical examples, where we show the performance of the model and developed numerical schemes.

2. The approach and auxiliary results

Our numerical analysis follows by first proving the existence of a solution (u_h, θ_h) on a possibly discretization-dependent time interval $[0, t_h^*]$ and then extending the existence to $[0,T]$ by means of a suitable uniform bound on this solution. This approach is in the general spirit of, e.g., [1, 6, 7], which have investigated single-physics wave models. The focal and most delicate point of the numerical analysis here is the derivation of a suitable energy bound for the nonlinear wave-heat system. To this end, since the acoustic component is strongly damped, the idea is to rewrite the semi-discrete problem in the following parabolic form:

$$\begin{cases} \partial_t^2 u_h - \beta_0 \Delta_h \partial_t u_h = q(\theta)\Delta_h u_h + \tilde{\beta}(\theta_h)\Delta_h \partial_t u_h - \mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h) + f_h, \\ \partial_t \theta_h - \kappa \Delta_h \theta_h + \nu \theta_h = \mathcal{Q}(u_h, \partial_t u_h, \theta_h), \end{cases} \tag{2.1}$$

with $\tilde{\beta}(\theta_h) \coloneqq \beta(\theta_h) - \beta_0$. This setup will allow us to exploit, to a certain extent, estimates for semi-discrete parabolic problems. For this reason, we present next two estimates for linear parabolic problems that will be used in Sections 4 and 5.

2.1. A maximal regularity estimate for linear parabolic problems.

Given $g_h \in L^2(0,T;V_h)$, $b_0 > 0$ and $\nu \geq 0$, consider the problem

$$(\partial_t w_h, \varphi_h)_{L^2} + b_0 a(w_h, \varphi_h) + \nu (w_h, \varphi_h)_{L^2} = (g_h, \varphi_h)_{L^2}, \qquad \forall \varphi_h \in V_h. \tag{2.2}$$

The finite element analysis of this problem with homogeneous initial data using a maximal L^p regularity approach can be found, for example, in [1, Theorem 1.1]. For completeness, we present here the derivation of the L^2-based energy bound, where compared to [1] we allow for non-zero initial data.

Lemma 2.1. *Let $g_h \in L^2(0,T;V_h)$. The solution of (2.2) satisfies*

$$\begin{aligned} \frac{1}{2}\int_0^t \|\partial_t w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \Big(\frac{b_0}{2}+1\Big)\|\nabla w_h\|_{L^2(\Omega)}^2\Big|_0^t + \frac{\nu}{2}\|w_h\|_{L^2(\Omega)}^2\Big|_0^t \\ + \frac{b_0}{2}\int_0^t \|\Delta_h w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \nu \int_0^t \|\nabla w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s \\ \leq \frac{1}{2}\Big(1+\frac{1}{b_0}\Big)\int_0^t \|g_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s. \end{aligned} \tag{2.3}$$

Proof. We test (2.2) with $\varphi_h = \partial_t w_h$ and integrate over $(0,t)$ for $t \in (0,T)$ to obtain

$$\int_0^t \|\partial_t w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \frac{b_0}{2}\|\nabla w_h\|_{L^2(\Omega)}^2\Big|_0^t + \frac{\nu}{2}\|w_h\|_{L^2(\Omega)}^2\Big|_0^t = \int_0^t (g_h(s), \partial_t w_h(s))_{L^2}\,\mathrm{d}s.$$

By choosing instead $\varphi_h = -\Delta_h w_h$, we obtain

$$\|\nabla w_h\|_{L^2(\Omega)}^2\Big|_0^t + b_0\int_0^t \|\Delta_h w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \nu\int_0^t \|\nabla w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s = \int_0^t (g_h(s), \Delta_h w_h(s))_{L^2}\,\mathrm{d}s.$$

Adding the estimates and employing Young's inequality yields (2.3). □

The bound in Lemma 2.2 will be employed in the error analysis of the semi-discrete wave subproblem in Section 4 with $w_h = \partial_t(\mathrm{R}_h u - u_h)$.

2.2. Additional estimate for a more regular right-hand side.

When working with the semi-discrete heat equation, we will have a relatively regular in time right-hand side due to the properties of the semi-discrete pressure field. For this reason, we also derive here an additional bound for parabolic problems that assumes more regularity in time of the right-hand side.

Lemma 2.2. *Let $g_h \in H^1(0,T;V_h)$. Then the solution of (2.2) satisfies*

$$\begin{aligned} \|\partial_t w_h(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla \partial_t w_h(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + \frac{b_0}{2}\|\Delta_h w_h(t)\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\|\nabla w_h(t)\|_{L^2(\Omega)}^2 \\ \lesssim e^{CT}(\|g_h\|_{H^1(L^2(\Omega))}^2 + \|\Delta_h w_h(0)\|_{L^2(\Omega)}^2 + \|\nabla w_h(0)\|_{L^2(\Omega)}^2). \end{aligned} \tag{2.4}$$

Proof. Note that $g_h \in H^1(0,T;V_h) \hookrightarrow C([0,T];V_h)$. Testing with $\varphi_h = -\Delta_h \partial_t w_h$ and integrating in time leads to

$$\int_0^t \|\nabla \partial_t w_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{b_0}{2}\|\Delta_h w_h\|^2_{L^2(\Omega)}\Big|_0^t + \frac{\nu}{2}\|\nabla w_h\|^2_{L^2(\Omega)}\Big|_0^t = -\int_0^t (g_h(s), \Delta_h \partial_t w_h(s))_{L^2} \,\mathrm{d}s.$$

To treat the right-hand side, we integrate by parts in time:

$$-\int_0^t (g_h(s), \Delta_h \partial_t w_h(s))_{L^2} \,\mathrm{d}s = -(g_h(s), \Delta_h w_h(s))_{L^2}\Big|_0^t + \int_0^t (\partial_t g_h(s), \Delta_h w_h(s))_{L^2} \,\mathrm{d}s.$$

Using Young's inequality, we have for any $\varepsilon > 0$

$$\begin{aligned} &\int_0^t \|\nabla \partial_t w_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{b_0}{2}\|\Delta_h w_h\|^2_{L^2(\Omega)}\Big|_0^t + \frac{\nu}{2}\|\nabla w_h\|^2_{L^2(\Omega)}\Big|_0^t \\ &\lesssim \|g_h(t)\|^2_{L^2(\Omega)} + \varepsilon\|\Delta w_h(t)\|^2_{L^2(\Omega)} + \|g_h(0)\|^2_{L^2(\Omega)} + \|\Delta_h w_h(0)\|^2_{L^2(\Omega)} \\ &\qquad + \int_0^t \|\partial_t g_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \int_0^t \|\Delta_h w_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s. \end{aligned}$$

Then by choosing ε sufficiently small and employing Grönwall's inequality, we obtain

$$\begin{aligned} &\int_0^t \|\nabla \partial_t w_h(s)\|^2_{L^2(\Omega)} \,\mathrm{d}s + \frac{b_0}{2}\|\Delta_h w_h(t)\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|\nabla w_h(t)\|^2_{L^2(\Omega)} \\ &\lesssim e^{CT}(\|g_h\|^2_{H^1(L^2(\Omega))} + \|\Delta_h w_h(0)\|^2_{L^2(\Omega)} + \|\nabla w_h(0)\|^2_{L^2(\Omega)}). \end{aligned}$$

Additionally, since $\partial_t w_h = b_0 \Delta_h w_h - \nu w_h + g_h$, we can bootstrap the above estimate to obtain

$$\begin{aligned} \|\partial_t w_h(t)\|^2_{L^2(\Omega)} &= \|b_0 \Delta_h w_h(t) - \nu w_h(t) + g_h(t)\|^2_{L^2(\Omega)} \\ &\lesssim e^{CT}(\|g_h\|^2_{H^1(L^2(\Omega))} + \|\Delta_h w_h(0)\|^2_{L^2(\Omega)} + \|\nabla w_h(0)\|^2_{L^2(\Omega)}). \end{aligned}$$

By adding the two bounds, we arrive at (2.4). □

The bound in Lemma 2.2 will be employed in the error analysis of the semi-discrete heat subproblem in Section 5 with $w_h = \mathrm{R}_h\theta - \theta_h$.

2.3. **Embeddings and inverse estimates.** When deriving estimates in Sections 4 and 5, we will utilize (discrete) embedding results and inverse estimates for finite element functions in the upcoming error analysis. In particular, the following embedding holds:

$$H^2(\Omega) \hookrightarrow W^{1,d+\delta}(\Omega) \hookrightarrow L^\infty(\Omega), \quad d \in \{1,2,3\},\ \delta > 0.$$

For $\phi_h \in V_h$, we have the discrete Sobolev embedding

$$\|\phi_h\|_{L^\infty(\Omega)} + \|\phi_h\|_{W^{1,6}(\Omega)} \leq C\|\Delta_h \phi_h\|_{L^2(\Omega)}, \tag{2.5}$$

where C is independent of h; see, for example, [1, 6, 9]. Furthermore, the following inverse estimates are used in the analysis:

$$\|\nabla \varphi_h\|_{L^2(\Omega)} \leq Ch^{-1}\|\varphi_h\|_{L^2(\Omega)}, \tag{2.6a}$$

$$\|\Delta_h \varphi_h\|_{L^2(\Omega)} \leq Ch^{-1}\|\nabla\varphi_h\|_{L^2(\Omega)}, \tag{2.6b}$$

$$\|\varphi_h\|_{L^\infty(\Omega)} \leq Ch^{-d/p}\|\varphi_h\|_{L^p(\Omega)}, \tag{2.6c}$$

for $\varphi_h \in V_h$ and $p \in [1,\infty]$, with constants independent of h.

We also recall the following bounds for the interpolant:

$$\|\varphi - \mathrm{I}_h\varphi\|_{L^p(\Omega)} + h\|\varphi - \mathrm{I}_h\varphi\|_{W^{1,p}(\Omega)} \leq Ch^{\ell+1}\|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in W^{\ell+1,p}(\Omega),$$

for $2 \leq p \leq \infty$ and $1 \leq \ell \leq k$.

2.4. **Properties of the Ritz projection.** In Sections 4 and 5, we also heavily rely on certain properties of the Ritz projection. We collect these results here for convenience. For the purpose of estimating the approximation errors in the upcoming analysis, we need the following approximation result for $0 \leq \ell \leq \eta$:

$$h\|\varphi - \mathrm{R}_h\varphi\|_{W^{1,p}(\Omega)} \leq Ch^{\ell+1}\|\varphi\|_{W^{\ell+1,p}(\Omega)}, \quad \varphi \in W^{\ell+1,p}(\Omega), \tag{2.7}$$

for all $2 \leq p \leq \infty$; see, for example, [4, Thm. 8.5.3]. In particular, we often employ the stability bound

$$\|\mathrm{R}_h\varphi\|_{L^\infty(\Omega)} \leq C\|\varphi\|_{W^{1,\infty}(\Omega)}.$$

Note that one could also replace the right-hand side with the $W^{1,p}$-norm for $p > d$, but for better readability, we employ the above bound. We further need the estimate

$$\|\mathrm{R}_h\varphi - \varphi\|_{L^\infty(\Omega)} \leq Ch^{1/2}\|\varphi\|_{H^2(\Omega)}, \tag{2.8}$$

which is obtained by inserting the nodal interpolation operator, using the inverse estimates (2.6), and the L^∞-estimate in [4, Theorem 4.4.20].

Lemma 2.3. *Let $\delta > 0$, $d \in \{1,2,3\}$, and $\mu \in C^{\eta+1}(\mathbb{R})$. Then, there is a constant $C > 0$, independent of h, such that for all $\phi_h \in V_h$ it holds*

$$\|\mathrm{R}_h[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)} \leq \|\mu(\psi_h)\|_{L^\infty(\Omega)}\|\phi_h\|_{L^2(\Omega)} + h^{\delta/(d+\delta)}C(\|\psi_h\|_{W^{1,d+\delta}(\Omega)})\|\phi_h\|_{L^2(\Omega)}.$$

Proof. The proof can be found in Appendix A. □

3. Existence and uniqueness on a discretization-dependent time interval

In this section, we show that the problem has a solution on a possibly h-dependent time interval $[0, t_h^*]$. In subsequent sections, we will carry out the estimates on this time interval with the goal of obtaining a uniform bound on (u_h, θ_h) in a suitable norm that will allow us to prolong the existence to $[0, T]$. As usual, we split the errors as follows:

$$\begin{aligned} u - u_h &= (u - \mathrm{R}_h u) + (\mathrm{R}_h u - u_h)\,, \\ \theta - \theta_h &= (\theta - \mathrm{R}_h\theta) + (\mathrm{R}_h\theta - \theta_h)\,, \end{aligned}$$

and denote the discrete errors by $e_h^u = \mathrm{R}_h u - u_h$ and $e_h^\theta = \mathrm{R}_h\theta - \theta_h$.

We aim to prove that the semi-discrete problem has a solution on the time interval $[0, t_h^*]$, where we define t_h^* as follows:

$$\begin{aligned} t_h^* \coloneqq \sup \Big\{ t \in (0,T] \mid\ & \text{a unique solution } (u_h, \theta_h) \in C^2([0,t]; V_h) \times C^1([0,t]; V_h) \\ & \text{of (1.7) and (1.8) exists and} \\ & h^{-1/2-\varepsilon}(\|\partial_t e_h^u(s)\|_{H^1(\Omega)} + \|\Delta_h e_h^u(s)\|_{L^2(\Omega)}) \leq C_0, \\ & h^{-1/2-\varepsilon}\|\Delta_h e_h^\theta(s)\|_{L^2(\Omega)} \leq C_0, \\ & \text{for all } s \in [0,t] \Big\} \end{aligned} \tag{3.1}$$

for some $\varepsilon \in (0, 1/2)$, and a constant $C_0 > 0$ independent of h. The particular choice of terms and norms involved in (3.1) is motivated by the needs of deriving the estimates, as will become apparent below and in Sections 4 and 5. The first claim of this section concerns the accuracy of the approximate initial data.

Lemma 3.1. *Under the assumptions of Theorem 1.2, with the approximate initial values chosen to be $(u_h(0), \partial_t u_h(0), \theta_h(0)) = (\mathrm{R}_h u_0, \mathrm{R}_h u_1, \mathrm{R}_h\theta_0)$, we have*

$$\|\partial_t e_h^u(0)\|_{H^1(\Omega)} + \|\Delta_h e_h^u(0)\|_{L^2(\Omega)} \leq Ch^\eta$$

and

$$\|\Delta_h e_h^\theta(0)\|_{L^2(\Omega)} \leq Ch^\eta.$$

Proof. With our choice of the initial data we immediately have $e_h^u(0) = \partial_t e_h^u(0) = e_h^\theta(0) = 0$, and thus the statement trivially holds. ☐

We next tackle the existence of a unique solution of the semi-discrete system on a possibly h-dependent time interval, which will allow us to conclude that $t_h^* > 0$.

Proposition 3.2. *Under the assumptions of Theorem 1.2, we have $t_h^* > 0$.*

Proof. The statement will follow by considering a first-order rewriting of the system and applying on it a local version of the Picard–Lindelöf theorem on the open set

$$U_h = \Big\{(u_h, \partial_t u_h, \theta_h) \in V_h^3 : \|k_{\mathrm{W}}(\theta_h)u_h\|_{L^\infty(\Omega)} + \|k_{\mathrm{K}}(\theta_h)\partial_t u_h\|_{L^\infty(\Omega)} < r + \delta\Big\} \tag{3.2}$$

with $\delta > 0$ to be determined below, and radius r from Theorem 1.2. To see that the initial values belong to U_h, we can uniformly bound $\theta_h(0)$ using the inverse estimate in (2.6c) and discrete embedding (2.5) as follows:

$$\begin{aligned}\|\theta_h(0) - \theta(0)\|_{L^\infty(\Omega)} &\lesssim h^{-d/6}\|\theta_h(0) - \mathrm{R}_h\theta(0)\|_{L^6(\Omega)} + \|\mathrm{R}_h\theta(0) - \theta(0)\|_{L^\infty(\Omega)} \\ &\lesssim h^{-d/6}\|\nabla(\theta_h(0) - \mathrm{R}_h\theta(0))\|_{L^2(\Omega)} + \|\mathrm{R}_h\theta(0) - \theta(0)\|_{L^\infty(\Omega)} \\ &\lesssim h^{-d/6}\|\Delta_h e_h^\theta(0)\|_{L^2(\Omega)} + h^{1/2}\|\theta\|_{L_t^\infty(H^2(\Omega))} \leq Ch^\varepsilon,\end{aligned} \tag{3.3}$$

where we have used Lemma 3.1 in the last step. Considering the $u_h(0)$ and $\partial_t u_h(0)$ terms, we similarly have

$$\begin{aligned}\|u_h(0) - u(0)\|_{L^\infty(\Omega)} &\lesssim h^{-d/6}\|\Delta_h e_h^u(0)\|_{L_t^\infty(L^2(\Omega))} + h^{1/2}\|u\|_{L_t^\infty(H^2(\Omega))} \leq Ch^\varepsilon, \\ \|\partial_t u_h(0) - u_t(0)\|_{L^\infty(\Omega)} &\lesssim h^{-d/6}\|\nabla\partial_t e_h^u(0)\|_{L^2(\Omega)} + h^{1/2}\|u_t\|_{L_t^\infty(H^2(\Omega))} \leq Ch^\varepsilon,\end{aligned} \tag{3.4}$$

using (2.8). Combining the three estimates yields

$$\begin{aligned}\|k_{\mathrm{W}}(\theta_h(0))u_h(0) - k_{\mathrm{W}}(\theta(0))u(0)\|_{L^\infty(\Omega)} &\leq Ch^\varepsilon, \\ \|k_{\mathrm{K}}(\theta_h(0))\partial_t u_h(0) - k_{\mathrm{K}}(\theta(0))u_t(0)\|_{L^\infty(\Omega)} &\leq Ch^\varepsilon,\end{aligned} \tag{3.5}$$

and thus $(u_h(0), \partial_t u_h(0), \theta_h(0)) \in U_h$ for any δ if $h \leq h_0$ is small enough.

To state the semi-discrete problem in a compact manner, we introduce the operator Λ_h defined by

$$(\Lambda_h(u_h, \partial_t u_h, \theta_h)\varphi_h, \psi_h)_{L^2} = ((1 + 2k_{\mathrm{W}}(\theta_h)u_h + 2k_{\mathrm{K}}(\theta_h)\partial_t u_h)\varphi_h, \psi_h)_{L^2}$$

for $\varphi_h, \psi_h \in V_h$. Further, given $\mu \in C^{0,1}(\mathbb{R})$, we introduce the operator

$$(A_h(\mu(\theta_h))\varphi_h, \psi_h)_{L^2} = (\nabla\varphi_h, \nabla(\mu(\theta_h)\psi_h))_{L^2}.$$

The time-differentiated semi-discrete problem can then be written as (with L^2-projection π_h)

$$\begin{aligned}\Lambda_h(u_h, \partial_t u_h, \theta_h)\partial_t^2 u_h =& A_h(q(\theta_h))u_h + A_h(\beta(\theta_h))\partial_t u_h \\ &- 2\pi_h\big(k_{\mathrm{W}}(\theta_h)\partial_t u_h^2 + \ell\nabla u_h \cdot \partial_t u_h\big) - f_h, \\ \partial_t\theta_h =& -\kappa\Delta_h\theta_h - \nu\theta_h + \pi_h\mathcal{Q}(u_h, \partial_t u_h, \theta_h).\end{aligned} \tag{3.6}$$

Note that the operator Λ_h is invertible on U_h for small enough r, since we can find $\gamma, \delta > 0$, independent of h, such that

$$1 + 2k_{\mathrm{W}}(\theta_h)u_h + 2k_{\mathrm{K}}(\theta_h)\partial_t u_h \geq \gamma > 0.$$

Therefore, the semi-discrete problem can be further rewritten as a first-order system for $\boldsymbol{v}_h = (u_h, \partial_t u_h, \theta_h)^T$:

$$\begin{cases} \partial_t \boldsymbol{v}_h = F(\boldsymbol{v}_h), \\ \boldsymbol{v}_h(0) = (u_{0h}, u_{1h}, \theta_{0h})^T, \end{cases} \tag{3.7}$$

where the right-hand side is given by

$$\begin{aligned} &F((u_h, \partial_t u_h, \theta_h)^T) \\ &= \Big((\Lambda_h(u_h, \partial_t u_h, \theta_h))^{-1} \left(A_h(q(\theta_h)) u_h + A_h(\beta(\theta_h)) \partial_t u_h - \pi_h\big(2k_{\mathrm{W}}(\theta_h)\partial_t u_h^2\big) \right. \\ &\qquad \left. - \pi_h\big(2\ell \nabla u_h \cdot \partial_t u_h\big) - f_h \right),\ \partial_t u_h,\ -\kappa \Delta_h \theta_h - \nu \theta_h + \pi_h \mathcal{Q}(u_h, \partial_t u_h, \theta_h) \Big)^T. \end{aligned} \tag{3.8}$$

Furthermore, system (3.7) has a locally Lipschitz continuous right-hand side (3.8). Indeed, Lipschitz continuity of the right-hand side follows by the fact that V_h is a finite-dimensional space in which we can use inverse estimates (2.6a)–(2.6b).

Thus by the local version of the Picard–Lindelöf theorem, a unique solution $(u_h, \theta_h) \in C^2([0,T]; V_h) \times C^1([0,T]; V_h)$ of (3.6), supplemented with approximate initial data, exists on $[0, \tilde{t}]$ for some $\tilde{t} > 0$. Since the initial errors in Lemma 3.1 in fact vanish, we have by the continuity and the equivalence of norms on V_h^3 that the errors e_h^u and e_h^θ still satisfy the bounds in (3.1) for a short time. Therefore, we conclude that $t_h^* > 0$. □

We also prove two uniform boundedness results on $[0, t_h^*]$ that will be useful in the next step of the error analysis.

Lemma 3.3. *Let the assumptions of Theorem 1.2 hold. Then the following bounds hold on* $[0, t_h^*]$*:*

$$\|u_h\|_{L_t^\infty(L^\infty(\Omega))} + \|\nabla u_h\|_{L_t^\infty(L^\infty(\Omega))} + \|\partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))} \lesssim 1$$

and

$$\|\theta_h\|_{L_t^\infty(L^\infty(\Omega))} + \|\theta_h\|_{L_t^\infty(W^{1,d+\delta}(\Omega))} + \|\partial_t \theta_h\|_{L_t^\infty(L^3(\Omega))} \lesssim 1.$$

Proof. The statement follows by a repeated use of the stability properties of the Ritz projection and inverse estimates (2.6a)–(2.6c). We have already shown in the previous proof that for $s \in [0, t_h^*]$

$$\|u_h(s)\|_{L^\infty(\Omega)} + \|\partial_t u_h(s)\|_{L^\infty(\Omega)} \lesssim 1.$$

Further, by (2.5), we have

$$\|\nabla u_h(s)\|_{L^\infty(\Omega)} \lesssim \|\mathrm{R}_h u(s)\|_{W^{1,\infty}(\Omega)} + \|e_h^u(s)\|_{W^{1,\infty}(\Omega)} \lesssim \|u(s)\|_{W^{1,\infty}(\Omega)} + h^{-d/6}\|\Delta_h e_h^u(s)\|_{L^2(\Omega)}.$$

Similarly, for $d + \delta \leq 6$, by (2.5), we have

$$\|\theta_h(s)\|_{L^\infty(\Omega)} \leq \|\mathrm{R}_h \theta(s)\|_{W^{1,d+\delta}(\Omega)} + \|e_h^\theta(s)\|_{W^{1,d+\delta}(\Omega)} \lesssim \|\mathrm{R}_h \theta(s)\|_{W^{1,d+\delta}(\Omega)} + \|\Delta_h e_h^\theta(s)\|_{L^2(\Omega)}.$$

Lastly, inserting the equation (2.1) for θ_h, the stability of the L^2-projection π_h in L^3, and the relation $\Delta_h \mathrm{R}_h = \pi_h \Delta$, we obtain

$$\begin{aligned} \|\partial_t \theta_h(s)\|_{L^3(\Omega)} &\lesssim \|\Delta_h \theta_h\|_{L^3(\Omega)} + \|\theta_h\|_{L^3(\Omega)} + \|\mathcal{Q}(u_h, \partial_t u_h, \theta_h)\|_{L^3(\Omega)} \\ &\lesssim h^{-d/6}\|\Delta_h e_h^\theta(s)\|_{L^2(\Omega)} + \|\pi_h \Delta \theta\|_{L^3(\Omega)} + \|\theta_h\|_{L^3(\Omega)} \\ &\qquad + C(\|u_h(s)\|_{L^\infty(\Omega)}, \|\partial_t u_h(s)\|_{L^\infty(\Omega)}, \|\theta_h(s)\|_{L^\infty(\Omega)}) \lesssim 1\,. \end{aligned}$$

The definition of t_h^* in (3.1) closes the proof. □

As a corollary of Lemma 3.3, on account of the assumptions made on the temperature-dependent medium parameters, we also have the following uniform bounds.

Corollary 3.4. *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned}&\|\tilde{\beta}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} + \|q(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}\\ &\quad + \|k_{\mathrm{W}}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} + \|\alpha(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} \lesssim 1\end{aligned}$$

on $[0, t_h^*]$.

In the next step of the *a priori* error analysis, we wish to derive a uniform estimate for (u_h, θ_h) that will allow us to show that

$$(u_h(t_h^*), \partial_t u_h(t_h^*), \theta_h(t_h^*)) \in U_h,$$

which will lead us to the conclusion that (u_h, θ_h) exists on $[0, T]$. We focus first on the acoustic subproblem and estimating $e_h^u = \mathrm{R}_h u - u_h$.

4. Estimates for the semi-discrete wave subproblem

In this section, our aim is to derive an energy estimate for $e_h^u = \mathrm{R}_h u - u_h$ on $[0, t_h^*]$ with t_h^* defined in (3.1). Toward estimating e_h^u, we observe first that the Ritz projection of u satisfies

$$\begin{aligned}&(\partial_t^2 \mathrm{R}_h u, \phi_h)_{L^2} + \beta_0 a(\partial_t \mathrm{R}_h u, \phi_h)\\ &= (\Delta u, q(\theta)\phi_h)_{L^2} + (\Delta \partial_t u, \tilde{\beta}(\theta)\phi_h)_{L^2}\\ &\quad + (\mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta), \phi) + (f, \phi_h)_{L^2} + (\delta^u, \phi_h)_{L^2},\end{aligned}$$

where the defect is given by

$$\begin{aligned}&(\delta^u, \phi_h)_{L^2}\\ &= (\partial_t^2 \mathrm{R}_h u - u_{tt}, \phi_h)_{L^2}\\ &\quad - (\mathcal{N}(u, u_t, u_{tt}, \nabla u, \nabla u_t, \theta) - \mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta), \phi_h)_{L^2}.\end{aligned}$$

We can estimate the defect using the following result.

Lemma 4.1. *Under the assumptions of Theorem 1.2, the following estimate holds:*

$$\|\delta^u\|_{L^2(L^2(\Omega))} \leq C(\|u\|_{\mathcal{X}_u}) h^\eta.$$

Proof. The proof follows by rewriting the difference of the $\mathcal{N}$ terms as follows:

$$\begin{aligned}&\mathcal{N}(u, u_t, u_{tt}, \nabla u, \nabla u_t, \theta) - \mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta)\\ &= 2k_{\mathrm{W}}(\theta)((u - \mathrm{R}_h u)u_{tt} + \mathrm{R}_h u(u_{tt} - \partial_t^2 \mathrm{R}_h u)) + 2k_{\mathrm{W}}(\theta)(u_t - \partial_t \mathrm{R}_h u)(u_t + \partial_t \mathrm{R}_h u)\\ &\quad + 2k_{\mathrm{K}}(\theta)((u_t - \partial_t \mathrm{R}_h u)u_{tt} + \partial_t \mathrm{R}_h u(u_{tt} - \partial_t^2 \mathrm{R}_h u))\\ &\quad + 2\ell \nabla (u - \mathrm{R}_h u) \cdot \nabla u_t + 2\ell \nabla \mathrm{R}_h u \cdot \nabla (u_t - \partial_t \mathrm{R}_h u)\end{aligned}$$

and using the properties of the Ritz projection. Indeed, we have

$$\begin{aligned}&\|2k_{\mathrm{W}}(\theta)((u - \mathrm{R}_h u)u_{tt} + \mathrm{R}_h u(u_{tt} - \partial_t^2 \mathrm{R}_h u)) + 2k_{\mathrm{W}}(\theta)(u_t - \partial_t \mathrm{R}_h u)(u_t + \partial_t \mathrm{R}_h u)\|_{L^2(L^2(\Omega))}\\ &\lesssim \|k_{\mathrm{W}}(\theta)\|_{L^\infty(L^\infty(\Omega))}\|u - \mathrm{R}_h u\|_{L^\infty(L^2(\Omega))}\|u_{tt}\|_{L^2(L^\infty(\Omega))} + \|\mathrm{R}_h u\|_{L^\infty(L^\infty(\Omega))}\|u_{tt} - \partial_t^2 \mathrm{R}_h u\|_{L^2(L^2(\Omega))}\\ &\quad + \|k_{\mathrm{W}}(\theta)\|_{L^\infty(L^\infty(\Omega))}\|u_t - \partial_t \mathrm{R}_h u\|_{L^2(L^4(\Omega))}\|u_t + \partial_t \mathrm{R}_h u\|_{L^2(L^4(\Omega))}.\end{aligned}$$

Next,

$$\begin{aligned}&\|2k_{\mathrm{K}}(\theta)((u_t - \partial_t \mathrm{R}_h u)u_{tt} + \partial_t \mathrm{R}_h u(u_{tt} - \partial_t^2 \mathrm{R}_h u))\|_{L^2(L^2(\Omega))}\\ &\lesssim \|k_{\mathrm{K}}(\theta)\|_{L^\infty(L^\infty(\Omega))}\|u_t - \partial_t \mathrm{R}_h u\|_{L^\infty(L^2(\Omega))}\|u_{tt}\|_{L^2(L^\infty(\Omega))} + \|\partial_t \mathrm{R}_h u\|_{L^\infty(L^\infty(\Omega))}\|u_{tt} - \partial_t^2 \mathrm{R}_h u\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Finally,

$$\begin{aligned}&\|\ell \nabla (u - \mathrm{R}_h u) \cdot \nabla u_t + \ell \nabla \mathrm{R}_h u \cdot \nabla (u_t - \partial_t \mathrm{R}_h u)\|_{L^2(L^2(\Omega))}\\ &\lesssim \|\nabla (u - \mathrm{R}_h u)\|_{L^2(L^2(\Omega))}\|\nabla u_t\|_{L^\infty(L^\infty(\Omega))} + \|\nabla \mathrm{R}_h u\|_{L^\infty(L^\infty(\Omega))}\|u_t - \partial_t \mathrm{R}_h u\|_{L^2(L^2(\Omega))}.\end{aligned}$$

Combining the bounds and relying on (2.7) leads to the claim. □

Hence, the error $e_h^u = \mathrm{R}_h u - u_h$ satisfies the following parabolic problem:

$$(\partial_t^2 e_h^u, \phi_h)_{L^2} + \beta_0 a(\partial_t e_h^u, \phi_h) = (\mathcal{F}_h^u, \phi_h)_{L^2},$$

with the right-hand side given by

$$\begin{aligned}(\mathcal{F}_h^u, \phi_h)_{L^2} \\ = (\delta^u, \phi_h)_{L^2} + (f - f_h, \phi_h)_{L^2} + (\Delta u, q(\theta)\phi_h)_{L^2} - (\Delta_h u_h, \mathrm{R}_h[q(\theta_h)\phi_h])_{L^2} \\ + (\Delta u_t, \tilde{\beta}(\theta)\phi_h)_{L^2} - (\Delta_h \partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2} \\ + (\mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h) - \mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta), \phi_h).\end{aligned} \tag{4.1}$$

We wish to apply the maximal regularity estimate result of Lemma 2.1 to this problem. Toward estimating the right-hand side, we can use Lemma 4.1 to bound δ^u. The next result will allow us to estimate the difference of q and $\tilde{\beta}$ terms in (4.1).

Lemma 4.2. *Under the assumptions of Theorem 1.2, the following bounds hold on* $[0, t_h^*]$*:*

$$\begin{aligned}&\sup_{\|\phi_h\|_{L^2(\Omega)}=1} |(\Delta u_t, \tilde{\beta}(\theta)\phi_h)_{L^2} - (\Delta_h \partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2}| \\ &\lesssim h^\eta + \|\Delta_h \partial_t e_h^u\|_{L^2(\Omega)} \big(\|\tilde{\beta}(\theta_h)\|_{L^\infty(\Omega)} + o(1)\big) + \|\tilde{\beta}(\theta) - \tilde{\beta}(\theta_h)\|_{W^{1,d+\delta}(\Omega)},\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}&\sup_{\|\phi_h\|_{L^2(\Omega)}=1} |(\Delta u, q(\theta)\phi_h)_{L^2} - (\Delta_h \partial_t u_h, \mathrm{R}_h[q(\theta_h)\phi_h])_{L^2}| \\ &\lesssim h^\eta + \|\Delta_h e_h^u\|_{L^2(\Omega)} \big(\|q(\theta_h)\|_{L^\infty(\Omega)} + o(1)\big) + \|q(\theta) - q(\theta_h)\|_{W^{1,d+\delta}(\Omega)},\end{aligned} \tag{4.3}$$

where the hidden constants are independent of h and t_h^*.

Proof. We only prove (4.2) as estimate (4.3) follows analogously. We first have the following rewriting:

$$\begin{aligned}&(\Delta u_t, \tilde{\beta}(\theta)\phi_h)_{L^2} - (\Delta_h \partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2} \\ &= a(\partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h]) - a(u_t, \tilde{\beta}(\theta)\phi_h) \\ &= a(\partial_t u_h - \mathrm{R}_h u_t, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h]) + a(\mathrm{R}_h u_t, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h]) - a(\mathrm{R}_h u_t, \tilde{\beta}(\theta)\phi_h) \\ &\qquad + a(\mathrm{R}_h u_t - u_t, \tilde{\beta}(\theta)\phi_h) \\ &= a(\partial_t e_h^u, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h]) + a(\mathrm{R}_h u_t, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h - \tilde{\beta}(\theta)\phi_h]) + a(\mathrm{R}_h u_t - u_t, (\mathrm{I} - \mathrm{R}_h)\tilde{\beta}(\theta)\phi_h) \\ &= -(\Delta_h \partial_t e_h^u, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2} - (\Delta_h \mathrm{R}_h u_t, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h - \tilde{\beta}(\theta)\phi_h])_{L^2} \\ &\qquad + a(\mathrm{R}_h u_t - u_t, (\mathrm{I} - \mathrm{R}_h)[\tilde{\beta}(\theta)\phi_h]),\end{aligned}$$

where we used the orthogonality of R_h in the a-inner product. Thus, employing Lemma 2.3 yields

$$\begin{aligned}&\sup_{\|\phi_h\|_{L^2(\Omega)}=1} |(\Delta u_t, \tilde{\beta}(\theta)\phi_h)_{L^2} - (\Delta_h \partial_t u_h, \mathrm{R}_h[\tilde{\beta}(\theta_h)\phi_h])_{L^2}| \\ &\le \|\Delta_h \partial_t e_h^u\|_{L^2(\Omega)} \big(\|\tilde{\beta}(\theta_h)\|_{L^\infty(\Omega)} + o(1)\big) + \|\Delta \partial_t u\|_{L^2(\Omega)} \|\tilde{\beta}(\theta) - \tilde{\beta}(\theta_h)\|_{W^{1,d+\delta}(\Omega)} \\ &\qquad + h^\eta \|u_t\|_{H^{\eta+1}(\Omega)} \sup_{\|\phi_h\|_{L^2(\Omega)}=1} \|(\mathrm{I} - \mathrm{R}_h)[\tilde{\beta}(\theta)\phi_h]\|_{H^1(\Omega)}.\end{aligned}$$

By the best approximation property,

$$\|(\mathrm{I} - \mathrm{R}_h)[\tilde{\beta}(\theta)\phi_h]\|_{H^1(\Omega)} \le \|(\mathrm{I} - \mathrm{I}_h)[\tilde{\beta}(\theta)\phi_h]\|_{H^1(\Omega)} \lesssim \|\tilde{\beta}(\theta)\|_{W^{\eta+1,\infty}(\Omega)} \|\phi_h\|_{L^2(\Omega)},$$

where the last estimate can be obtained analogously to the proof of [6, Lemma 5.2]. Thus, we have (4.2). ☐

Thanks to Lemma 4.2, we have

$$
\begin{aligned}
&\|\mathcal{F}_h^u\|_{L_t^2(L^2(\Omega))}\\
&\lesssim C(\|u\|_{\mathcal{X}_u})h^\eta + \|\Delta_h \partial_t e_h^u\|_{L_t^2(L^2(\Omega))}\big(\|\tilde{\beta}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))} + o(1)\big) + \|\tilde{\beta}(\theta) - \tilde{\beta}(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))}\\
&\qquad + \|\Delta_h e_h^u\|_{L_t^2(L^2(\Omega))}\big(\|q(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))} + o(1)\big) + \|q(\theta) - q(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))}\\
&\quad + \|\mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h) - \mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta\|_{L_t^2(L^2(\Omega))}.
\end{aligned}
$$

In the next step, we bound the difference of the $\mathcal{N}$ terms.

Lemma 4.3. *Under the assumptions of Theorem 1.2, the following estimate holds on* $[0, t_h^*]$*:*

$$
\begin{aligned}
&\|\mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h) - \mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta)\|_{L_t^2(L^2(\Omega))}\\
&\leq C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta})\Big\{\|k_\mathrm{W}(\theta) - k_\mathrm{W}(\theta_h))\|_{L_t^2(L^\infty(\Omega))} + \|k_\mathrm{K}(\theta) - k_\mathrm{K}(\theta_h))\|_{L_t^2(L^\infty(\Omega))}\\
&\quad + \|k_\mathrm{W}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))}\|e_h^u\|_{L_t^2(L^\infty(\Omega))} + \|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|k_\mathrm{K}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))}\|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))}\\
&\qquad + \|\nabla e_h^u\|_{L_t^2(L^2(\Omega))} + \|\nabla \partial_t e_h^u\|_{L_t^2(L^2(\Omega))}\Big\}\\
&\qquad + \Big\{\|k_\mathrm{W}(\theta_h)\, u_h\|_{L_t^\infty(L^\infty(\Omega))} + \|k_\mathrm{K}(\theta_h)\, \partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))}\Big\}\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))},
\end{aligned}
\tag{4.4}
$$

where the constant is independent of h and t_h^*.

Proof. We first rewrite the difference of $\mathcal{N}$ terms as follows:

$$
\begin{aligned}
&\mathcal{N}(\mathrm{R}_h u, \partial_t \mathrm{R}_h u, \partial_t^2 \mathrm{R}_h u, \nabla \mathrm{R}_h u, \nabla \partial_t \mathrm{R}_h u, \theta) - \mathcal{N}(u_h, \partial_t u_h, \partial_t^2 u_h, \nabla u_h, \nabla \partial_t u_h, \theta_h)\\
&= (k_\mathrm{W}(\theta) - k_\mathrm{W}(\theta_h))((\mathrm{R}_h u)^2)_{tt} + k_\mathrm{W}(\theta_h)((\mathrm{R}_h u)^2 - u_h^2)_{tt}\\
&\quad + (k_\mathrm{K}(\theta) - k_\mathrm{W}(\theta_h))((\mathrm{R}_h u_t)^2)_t + k_\mathrm{K}(\theta_h)((\mathrm{R}_h u_t)^2 - \partial_t u_h^2)_t + 2\ell \nabla e_h^u \cdot \nabla \mathrm{R}_h u_t\\
&\qquad + 2\ell \nabla u_h \cdot \nabla \partial_t e_h^u.
\end{aligned}
\tag{4.5}
$$

Using the fact that

$$
\begin{aligned}
&((\mathrm{R}_h u)^2 - u_h^2)_{tt}\\
&= 2(\mathrm{R}_h u - u_h)\partial_t^2 \mathrm{R}_h u + 2u_h(\partial_t^2 \mathrm{R}_h u - \partial_t^2 u_h) + 2(\partial_t \mathrm{R}_h u - \partial_t u_h)(\partial_t \mathrm{R}_h u + \partial_t u_h),
\end{aligned}
$$

we have by Hölder's inequality

$$
\begin{aligned}
&\|(k_\mathrm{W}(\theta) - k_\mathrm{W}(\theta_h))((\mathrm{R}_h u)^2)_{tt} + k_\mathrm{W}(\theta_h)((\mathrm{R}_h u)^2 - u_h^2)_{tt}\|_{L_t^2(L^2(\Omega))}\\
&\lesssim \|k_\mathrm{W}(\theta) - k_\mathrm{W}(\theta_h))\|_{L_t^2(L^\infty(\Omega))}\|((\mathrm{R}_h u)^2)_{tt}\|_{L_t^\infty(L^2(\Omega))}\\
&\quad + \|k_\mathrm{W}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))}\|e_h^u\|_{L_t^2(L^\infty(\Omega))}\|\partial_t^2 \mathrm{R}_h u\|_{L^\infty(L^2(\Omega))} + \|k_\mathrm{W}(\theta_h)\, u_h\|_{L_t^\infty(L^\infty(\Omega))}\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))}\\
&\qquad + \|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))}(\|\partial_t \mathrm{R}_h u\|_{L^\infty(L^\infty(\Omega))} + \|\partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))})),
\end{aligned}
$$

we note that $\|k_\mathrm{W}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))} \lesssim 1$ thanks to Corollary 3.4. Next, since

$$
((\mathrm{R}_h u_t)^2 - \partial_t u_h^2)_t = 2(\mathrm{R}_h u_t - \partial_t u_h)\partial_t^2 \mathrm{R}_h u + \partial_t u_h(\partial_t^2 \mathrm{R}_h u - u_{tt})
$$

we have

$$
\begin{aligned}
&\|(k_\mathrm{K}(\theta) - k_\mathrm{K}(\theta_h))((\mathrm{R}_h u_t)^2)_t + k_\mathrm{K}(\theta_h)((\mathrm{R}_h u_t)^2 - \partial_t u_h^2)_t\|_{L_t^2(L^2(\Omega))}\\
&\lesssim \|k_\mathrm{K}(\theta) - k_\mathrm{K}(\theta_h)\|_{L_t^2(L^\infty(\Omega))}\|((\mathrm{R}_h u_t)^2)_t\|_{L^\infty(L^2(\Omega))}\\
&\quad + \|k_\mathrm{K}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))}\|\nabla \partial_t e_h^u\|_{L_t^\infty(L^2(\Omega))}\|\partial_t^2 \mathrm{R}_h u\|_{L^2(L^3(\Omega))}\\
&\qquad + \|k_\mathrm{K}(\theta_h)\, \partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))}\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))}
\end{aligned}
$$

where we have relied on the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. We can further employ the fact that $\|k_{\mathrm{K}}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} \lesssim 1$. Lastly, we can estimate the gradient terms on the right-hand side of (4.5) as follows:

$$\begin{aligned}&\|\ell \nabla e^u_h \cdot \nabla \mathrm{R}_h u_t + \ell \nabla u_h \cdot \nabla \partial_t e^u_h\|_{L^2_t(L^2(\Omega))}\\ \lesssim& \|\nabla e^u_h\|_{L^2_t(L^2(\Omega))}\|\nabla \mathrm{R}_h u_t\|_{L^\infty(L^\infty(\Omega))} + \|\nabla u_h\|_{L^\infty_t(L^\infty(\Omega))}\|\nabla \partial_t e^u_h\|_{L^2_t(L^2(\Omega))},\end{aligned}$$

where we recall that $\|\nabla u_h\|_{L^\infty_t(L^\infty(\Omega))} \lesssim 1$ on $[0, t^*_h]$ thanks to Lemma 3.3. Combining the derived bounds leads to (4.4). ☐

We have now all the ingredients to estimate $\mathcal{F}^u_h$.

Lemma 4.4. *Under the assumptions of Theorem 1.2, we have*

$$\begin{aligned}&\|\mathcal{F}^u_h\|_{L^2_t(L^2(\Omega))}\\ &\le C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta})\Big\{h^\eta + \|k_{\mathrm{W}}(\theta) - k_{\mathrm{W}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta) - k_{\mathrm{K}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))}\\ &\quad + \|k_{\mathrm{W}}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}\|e^u_h\|_{L^2_t(L^\infty(\Omega))} + \|\partial_t e^u_h\|_{L^2_t(L^2(\Omega))} + \|k_{\mathrm{K}}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}\|\partial_t e^u_h\|_{L^2_t(L^2(\Omega))}\\ &\qquad + \|\nabla e^u_h\|_{L^2_t(L^2(\Omega))} + \|\nabla \partial_t e^u_h\|_{L^2_t(L^2(\Omega))} + \|\Delta_h e^u_h\|_{L^2_t(L^2(\Omega))}\Big\}\\ &\qquad + \|q(\theta) - q(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))} + \|\tilde{\beta}(\theta) - \tilde{\beta}(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))}\\ &\qquad + \|\Delta_h \partial_t e^u_h\|_{L^2_t(L^2(\Omega))}\big(\|\tilde{\beta}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))} + o(1)\big)\\ &\qquad + \big(\|k_{\mathrm{W}}(\theta_h)\, u_h\|_{L^\infty_t(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta_h)\, \partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}\big)\|\partial_t^2 e^u_h\|_{L^2_t(L^2(\Omega))}.\end{aligned}$$

Proof. The estimate follows by combining the results of Lemmas 4.1, 4.2, and 4.3. ☐

By employing the maximal regularity estimate of Lemma 2.1 with the choice $w_h = \partial_t e^u_h$, we obtain

$$\|\partial_t^2 e^u_h\|_{L^2_t(L^2(\Omega))} + \|\Delta_h \partial_t e^u_h\|_{L^2_t(L^2(\Omega))} + \|\nabla \partial_t e^u_h\|_{L^\infty_t(L^2(\Omega))} \lesssim \|\mathcal{F}^u_h\|_{L^2_t(L^2(\Omega))} + \|\nabla \partial_t e^u_h(0)\|_{L^2(\Omega)}.$$

Note that we can also bound $\|\Delta_h e^u_h\|_{L^\infty_t(L^2(\Omega))}$ by using

$$\|\Delta_h e^u_h\|_{L^\infty_t(L^2(\Omega))} \lesssim \|\Delta_h e^u_h(0)\|_{L^2(\Omega)} + \|\partial_t \Delta_h e^u_h\|_{L^2_t(L^2(\Omega))}.$$

Since by our choice of discrete initial data the terms at zero vanish, we have

$$\begin{aligned}&\|\partial_t^2 e^u_h\|_{L^2_t(L^2(\Omega))} + \|\Delta_h \partial_t e^u_h\|_{L^2_t(L^2(\Omega))} + \|\Delta_h e^u_h\|_{L^\infty_t(L^2(\Omega))}\\ &\qquad + \|\nabla \partial_t e^u_h\|_{L^\infty_t(L^2(\Omega))} \lesssim \|\mathcal{F}^u_h\|_{L^2_t(L^2(\Omega))}.\end{aligned} \tag{4.6}$$

This bound will be combined with an analogous one for the semi-discrete temperature equation, where we will look to either absorb the right-hand side terms by the left-hand side or handle them via Grönwall's inequality.

5. Estimates for the semi-discrete heat subproblem

In this section, we derive an estimate of e^θ_h on $[0, t^*_h]$ by suitably testing the semi-discrete heat subproblem. Recall that the heat equation in weak form is given by

$$(\theta_t, \phi)_{L^2} + \kappa a(\theta, \phi) + \nu(\theta, \phi)_{L^2} = (\alpha(\theta)(\zeta_1 u^2 + \zeta_2 u_t^2), \phi)_{L^2}$$

for all $\phi \in H^1_0(\Omega)$. The semi-discrete version is then given by

$$(\partial_t \theta_h, \phi_h)_{L^2} + \kappa a(\theta_h, \phi_h) + \nu(\theta_h, \phi_h)_{L^2} = (\alpha(\theta_h)(\zeta_1 u_h^2 + \zeta_2 \partial_t u_h^2), \phi_h)_{L^2}$$

for all $\phi_h \in V^\theta_h$. The Ritz projection of θ satisfies

$$\begin{aligned}&(\partial_t \mathrm{R}_h \theta, \phi_h)_{L^2} + \kappa a(\mathrm{R}_h \theta, \phi_h) + \nu(\mathrm{R}_h \theta, \phi_h)_{L^2}\\ &= (\alpha(\mathrm{R}_h \theta)(\zeta_1 (\mathrm{R}_h u)^2 + \zeta_2 (\partial_t \mathrm{R}_h u)^2), \phi_h)_{L^2} + (\delta^\theta, \phi_h)_{L^2}\end{aligned}$$

with the defect given by

$$(\delta^\theta, \phi_h)_{L^2} = (\partial_t \mathrm{R}_h\theta - \theta_t, \phi_h)_{L^2} + \nu(\mathrm{R}_h\theta - \theta, \phi_h)_{L^2} + (\alpha(\theta)(\zeta_1 u^2 + \zeta_2 u_t^2) - \alpha(\mathrm{R}_h\theta)(\zeta_1(\mathrm{R}_h u)^2 + \zeta_2(\partial_t \mathrm{R}_h u)^2), \phi_h)_{L^2}. \tag{5.1}$$

Thus, the error $e_h^\theta = \mathrm{R}_h\theta - \theta_h$ solves the parabolic problem

$$(\partial_t e_h^\theta, \phi_h)_{L^2} + \kappa a(e_h^\theta, \phi_h) + \nu(e_h^\theta, \phi_h)_{L^2} = (\mathcal{F}_h^\theta, \phi_h)_{L^2} \tag{5.2}$$

with the right-hand side

$$(\mathcal{F}_h^\theta, \phi_h)_{L^2} = (\delta^\theta, \phi_h)_{L^2} + (\alpha(\mathrm{R}_h\theta)(\zeta_1(\mathrm{R}_h u)^2 + \zeta_2(\partial_t \mathrm{R}_h u)^2) - \alpha(\theta_h)(\zeta_1 u_h^2 + \zeta_2 \partial_t u_h^2), \phi_h)_{L^2}.$$

Looking at the estimate of the acoustic right-hand side $\mathcal{F}_h^u$ in Lemma 4.4, we see that we have to further bound $\|q(\theta) - q(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))}$ and $\|\tilde\beta(\theta) - \tilde\beta(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))}$ in the course of the analysis of the semi-discrete heat equation. We intend to rely on the properties of the temperature-dependent speed of sound and sound diffusivity to conclude that

$$\|w(\theta) - w(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))} \lesssim \|\theta - \theta_h\|_{L_t^2(W^{1,d+\delta}(\Omega))}, \quad w \in \{q, \tilde\beta\}, \tag{5.3}$$

since $\|\theta\|_{L_t^\infty(W^{1,d+\delta}(\Omega))}, \|\theta_h\|_{L_t^\infty(W^{1,d+\delta}(\Omega))} \leq C$. Similarly, we will exploit the estimate

$$\begin{aligned} &\|k_{\mathrm{W}}(\theta) - k_{\mathrm{W}}(\theta_h)\|_{L_t^2(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta) - k_{\mathrm{K}}(\theta_h)\|_{L_t^2(L^\infty(\Omega))} \\ \lesssim\, &\|\theta - \theta_h\|_{L_t^2(L^\infty(\Omega))} \\ \lesssim\, &\|e_h^\theta\|_{L_t^2(L^\infty(\Omega))} + \|\theta - \mathrm{R}_h\theta\|_{L_t^2(L^\infty(\Omega))} \end{aligned} \tag{5.4}$$

since $\|\theta\|_{L_t^\infty(L^\infty(\Omega))}, \|\theta_h\|_{L_t^\infty(L^\infty(\Omega))} \leq C$. The error analysis of the heat equation should lead to bounds on $\|e_h^\theta\|_{L_t^\infty(L^\infty(\Omega))}$ and $\|e_h^\theta\|_{L_t^\infty(W^{1,d+\delta}(\Omega))}$ so that the terms $\|e_h^\theta\|_{L_t^2(L^\infty(\Omega))}$ and $\|e_h^\theta\|_{L_t^2(W^{1,d+\delta}(\Omega))}$ could be handled using Grönwall's inequality. These can be obtained via the embedding

$$\|e_h^\theta\|_{L_t^\infty(L^\infty(\Omega))} + \|e_h^\theta\|_{L_t^\infty(W^{1,d+\delta}(\Omega))} \lesssim \|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))}$$

and a suitable bound on $\|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))}$.

To this end, we plan to employ Lemma 2.2 on the semi-discrete heat subproblem, provided we have control of the right-hand side in the $H^1(0,t;L^2(\Omega))$ norm. We thus need to estimate $\|\mathcal{F}_h^\theta\|_{H_t^1(L^2(\Omega))}$ via bounds on $\|\mathcal{F}_h^\theta\|_{L_t^2(L^2(\Omega))}$ and $\|\partial_t\mathcal{F}_h^\theta\|_{L_t^2(L^2(\Omega))}$. We first estimate the defect term within $\mathcal{F}_h^\theta$ using the usual approximation properties of the Ritz projection. To improve the readability, we postpone proofs of the next two lemmas to the Appendix.

Lemma 5.1. *Under the assumptions of Theorem 1.2, the following estimate holds:*

$$\|\delta^\theta\|_{L_t^2(L^2(\Omega))} + \|\partial_t\delta^\theta\|_{L_t^2(L^2(\Omega))} \lesssim C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta}) h^{\eta+1}.$$

Proof. The proof is given in Appendix A. □

This result enables us to estimate $\mathcal{F}_h^\theta$ and $\partial_t\mathcal{F}_h^\theta$.

Lemma 5.2. *Under the assumptions of Theorem 1.2, we have the following two estimates on* $[0, t_h^*]$

$$\|\mathcal{F}_h^\theta\|_{L_t^2(L^2(\Omega))} \leq C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta}) \Big\{ h^{\eta+1} + \|e_h^u\|_{L_t^2(L^2(\Omega))} + \|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|e_h^\theta\|_{L_t^2(L^2(\Omega))} \Big\},$$

$$\begin{aligned} \|\partial_t\mathcal{F}_h^\theta\|_{L_t^2(L^2(\Omega))} \leq C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta}) \Big\{ & h^{\eta+1} + \|e_h^\theta\|_{L_t^2(L^2(\Omega))} + \|\partial_t e_h^\theta\|_{L_t^2(L^2(\Omega))} + \|\Delta_h e_h^\theta\|_{L_t^2(L^2(\Omega))} \\ & + \|e_h^u\|_{L_t^2(L^2(\Omega))} + \|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|\nabla\partial_t e_h^u\|_{L_t^2(L^2(\Omega))} \\ & + \|\alpha(\theta_h)\,\partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))} \|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} \Big\}, \end{aligned}$$

where the constants are independent of h *and* t_h^*.

Proof. The proof is given in Appendix A. ☐

From (5.2) via parabolic estimate (2.4) with the choice $w_h = e_h^\theta$, we have

$$\begin{aligned} &\|\partial_t e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\nabla e_h^\theta\|_{L_t^\infty(L^2(\Omega))} \\ &\lesssim \|\mathcal{F}_h^\theta\|_{H_t^1(L^2(\Omega))} + \|\Delta_h e_h^\theta(0)\|_{L^2(\Omega)} + \|\nabla e_h^\theta(0)\|_{L^2(\Omega)} = \|\mathcal{F}_h^\theta\|_{H_t^1(L^2(\Omega))}. \end{aligned} \tag{5.5}$$

on $[0, t_h^*]$. In the next section, we will combine this bound with the results of Section 4 to complete the proof Theorem 1.2.

6. Proof of the main result: *A priori* bounds for the coupled system

In this section, we prolong the existence of (u_h, θ_h) to $[0, T]$ and prove the main theoretical result of this work stated in Theorem 1.2.

6.1. **Uniform bound for the wave-heat system.** We first combine the two bounds for the semi-discrete pressure and heat subproblems to obtain the following result.

Proposition 6.1. *Let the assumptions of Theorem 1.2 hold. The following estimate holds:*

$$\begin{aligned} &\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h \partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h e_h^u\|_{L_t^\infty(L^2(\Omega))} + \|\partial_t e_h^u\|_{L_t^\infty(H^1(\Omega))} \\ &\quad + \|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\partial_t e_h^\theta\|_{L_t^\infty(L^2(\Omega))} \leq C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta}) h^\eta \end{aligned} \tag{6.1}$$

for $t \in [0, t_h^*]$, where the constant is independent of h and t_h^*.

Proof. Adding the two derived bounds (4.6) and (5.5) for the semi-discrete acoustic and heat subproblems leads to

$$\begin{aligned} &\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h \partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h e_h^u\|_{L_t^\infty(L^2(\Omega))} + \|\partial_t e_h^u\|_{L_t^\infty(H^1(\Omega))} \\ &\quad + \|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\partial_t e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\nabla e_h^\theta\|_{L_t^\infty(L^2(\Omega))} \\ &\lesssim \|\mathcal{F}_h^u\|_{L_t^2(L^2(\Omega))} + \|\mathcal{F}_h^\theta\|_{H_t^1(L^2(\Omega))}. \end{aligned}$$

On account of Lemmas 4.4 and 5.2, we then find that

$$\begin{aligned} &\|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h \partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|\Delta_h e_h^u\|_{L_t^\infty(L^2(\Omega))} + \|\nabla \partial_t e_h^u\|_{L_t^\infty(L^2(\Omega))} \\ &\quad + \|\Delta_h e_h^\theta\|_{L_t^\infty(L^2(\Omega))} + \|\partial_t e_h^\theta\|_{L_t^\infty(L^2(\Omega))} \\ &\lesssim C(\|u\|_{\mathcal{X}_u}, \|\theta\|_{\mathcal{X}_\theta}) \Big\{ h^\eta + \|k_{\mathrm{W}}(\theta) - k_{\mathrm{W}}(\theta_h))\|_{L_t^2(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta) - k_{\mathrm{K}}(\theta_h))\|_{L_t^2(L^\infty(\Omega))} \\ &\quad + \|e_h^u\|_{L_t^2(L^\infty(\Omega))} + \|\partial_t e_h^u\|_{L_t^2(L^2(\Omega))} + \|\nabla e_h^u\|_{L_t^2(L^2(\Omega))} + \|\nabla \partial_t e_h^u\|_{L_t^2(L^2(\Omega))} \\ &\quad + \|\Delta_h e_h^u\|_{L_t^2(L^2(\Omega))} + \|e_h^\theta\|_{L_t^2(L^2(\Omega))} + \|\partial_t e_h^\theta\|_{L_t^2(L^2(\Omega))} \Big\} \\ &\quad + \|q(\theta) - q(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))} + \|\tilde{\beta}(\theta) - \tilde{\beta}(\theta_h)\|_{L_t^2(W^{1,d+\delta}(\Omega))} + \mathcal{R}, \end{aligned} \tag{6.2}$$

where we have introduced the following short-hand notation:

$$\begin{aligned} \mathcal{R} = &\|\Delta_h \partial_t e_h^u\|_{L_t^2(L^2(\Omega))} \big(\|\tilde{\beta}(\theta_h)\|_{L_t^\infty(L^\infty(\Omega))} + o(1) \big) \\ &+ \|k_{\mathrm{W}}(\theta_h)\ u_h\|_{L_t^\infty(L^\infty(\Omega))} \|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} \\ &+ \|k_{\mathrm{K}}(\theta_h)\ \partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))} \|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))} \\ &+ \|\alpha(\theta_h)\ \partial_t u_h\|_{L_t^\infty(L^\infty(\Omega))} \|\partial_t^2 e_h^u\|_{L_t^2(L^2(\Omega))}. \end{aligned}$$

We first have, using (5.4),

$$\begin{aligned} &\|k_{\mathrm{W}}(\theta) - k_{\mathrm{W}}(\theta_h))\|_{L_t^2(L^\infty(\Omega))} + \|k_{\mathrm{K}}(\theta) - k_{\mathrm{K}}(\theta_h))\|_{L_t^2(L^\infty(\Omega))} \\ &\lesssim \|\Delta_h e_h^\theta\|_{L_t^2(L^2(\Omega))} + \|\theta - \mathrm{R}_h \theta\|_{L_t^2(L^\infty(\Omega))}. \end{aligned}$$

The approximation properties of the Ritz projection stated in (2.7) then yield

$$\begin{aligned}&\|k_{\mathrm{W}}(\theta)-k_{\mathrm{W}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))}\|+\|k_{\mathrm{K}}(\theta)-k_{\mathrm{K}}(\theta_h))\|_{L^2_t(L^\infty(\Omega))}\\ \lesssim\,&\|\Delta_h e^\theta_h\|_{L^2_t(L^2(\Omega))}+h^\eta\|\theta\|_{L^2(W^{\eta,\infty}(\Omega))}.\end{aligned}$$

The difference of q and $\tilde{\beta}$ terms can be further estimated as follows:

$$\begin{aligned}&\|q(\theta)-q(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))}+\|\tilde{\beta}(\theta)-\tilde{\beta}(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))}\\ \leq\,&C(\|\theta\|_{L^\infty_t(W^{1,d+\delta}(\Omega))},\|\theta_h\|_{L^\infty_t(W^{1,d+\delta}(\Omega))})(\|\theta-\theta_h\|_{L^2_t(W^{1,d+\delta}(\Omega))}\\ \leq\,&C(\|\theta\|_{L^\infty_t(W^{1,d+\delta}(\Omega))},\|\theta_h\|_{L^\infty_t(W^{1,d+\delta}(\Omega))})(\|e^\theta_h\|_{L^2_t(W^{1,d+\delta}(\Omega))}+\|\theta-\mathrm{R}_h\theta\|_{L^2_t(W^{1,d+\delta}(\Omega))}),\end{aligned}$$

and together with the discrete Sobolev embedding (2.5) (with $d+\delta\leq 6$) it holds

$$\|e^\theta_h\|_{L^2_t(W^{1,d+\delta}(\Omega))}\leq C\|\Delta_h e^\theta_h\|_{L^2_t(L^2(\Omega))}.$$

Again by the approximation properties of the Ritz projection stated in (2.7), we then have (with $d+\delta\geq 2$)

$$\begin{aligned}&\|q(\theta)-q(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))}+\|\tilde{\beta}(\theta)-\tilde{\beta}(\theta_h)\|_{L^2_t(W^{1,d+\delta}(\Omega))}\\ \lesssim\,&\|e^\theta_h\|_{L^2_t(W^{1,d+\delta}(\Omega))}+h^\eta\|\theta\|_{L^2(W^{1+\eta,d+\delta}(\Omega))},\end{aligned}$$

where we have also used the fact that $\|\theta\|_{L^\infty_t(W^{1,d+\delta}(\Omega))}$, $\|\theta_h\|_{L^\infty_t(W^{1,d+\delta}(\Omega))}\lesssim 1$.

It remains to discuss the terms within $\mathcal{R}$. Observe that these terms cannot be handled using Grönwall's inequality. Instead we rely on the smallness of

$$\|\tilde{\beta}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}+\|k_{\mathrm{W}}(\theta_h)u_h\|_{L^\infty_t(L^\infty(\Omega))}+\|k_{\mathrm{K}}(\theta_h)\partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}+\|\alpha(\theta_h)\partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}$$

to absorb them by the left-hand side of (6.2). This smallness can be achieved by the same computations as in (3.3) and (3.4) for time t instead of 0, and performing the estimate (3.5) also for $\tilde{\beta}(\theta_h)$ and $\alpha(\theta_h)\partial_t u_h$. In fact, by the smallness condition (1.9) of the exact solution, we obtain

$$\|\tilde{\beta}(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}+\|k_{\mathrm{W}}(\theta_h)u_h\|_{L^\infty_t(L^\infty(\Omega))}+\|k_{\mathrm{K}}(\theta_h)\partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}+\|\alpha(\theta_h)\partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}\leq r+Ch_0^\varepsilon.$$

Then by decreasing r and h_0, the e^u_h and e^θ_h terms within $\mathcal{R}$ can be absorbed by the left-hand side. An application of Grönwall's inequality thus yields

$$\begin{aligned}&\|\partial_t^2 e^u_h\|_{L^2_t(L^2(\Omega))}+\|\Delta_h\partial_t e^u_h\|_{L^2_t(L^2(\Omega))}+\|\Delta_h e^u_h\|_{L^\infty_t(L^2(\Omega))}+\|\nabla\partial_t e^u_h\|_{L^\infty_t(L^2(\Omega))}\\ &\quad+\|\Delta_h e^\theta_h\|_{L^\infty_t(L^2(\Omega))}+\|\partial_t e^\theta_h\|_{L^\infty_t(L^2(\Omega))}\leq C(\|u\|_{\mathcal{X}_u},\|\theta\|_{\mathcal{X}_\theta})h^\eta,\end{aligned}$$

as claimed. □

6.2. **Prolonging the interval of existence.** We are now ready for the final step in the well-posedness and error analysis, which will complete the proof of the main theoretical result of this work.

Proof of Theorem 1.2. Since the energy estimate (6.1) holds on $[0,t^*_h]$, we have

$$\begin{aligned}\|\Delta_h e^u_h(t^*_h)\|_{L^2(\Omega)}+\|\partial_t e^u_h(t^*_h)\|_{H^1(\Omega)}&\leq C(\|u\|_{\mathcal{X}_u},\|\theta\|_{\mathcal{X}_\theta})h^\eta\\ \|\Delta_h e^\theta_h(t^*_h)\|_{L^2(\Omega)}&\leq C(\|u\|_{\mathcal{X}_u},\|\theta\|_{\mathcal{X}_\theta})h^\eta.\end{aligned}$$

On account of $\eta\geq 1$, we can then guarantee that

$$\begin{aligned}\|\Delta_h e^u_h(t^*_h)\|_{L^2(\Omega)}+\|\partial_t e^u_h(t^*_h)\|_{H^1(\Omega)}&<C_0h^{1/2+\varepsilon},\\ \|\Delta_h e^\theta_h(t^*_h)\|_{L^2(\Omega)}&<C_0h^{1/2},\end{aligned}$$

provided h_0 is sufficiently small. Therefore, by the same estimate for time t^*_h as in (3.3), (3.4), and (3.5) $(u_h(t^*_h),\partial_t u_h(t^*_h),\theta_h(t^*_h))\in U_h$, where we recall that U_h was defined in (3.2). We can thus use the same reasoning from before but starting at the time $t=t^*_h$ to prolong the existence of solutions beyond t_h^*. The definition of t_h^* in (3.1) then implies $t_h^* = T$. Thus, the error estimate in (6.1) holds on $[0, T]$. This completes the proof. □

7. Numerical experiments

In this section, we explore fully discrete numerical approximations of the pressure-temperature system (1.1) and present numerical examples in two-dimensional spatial domains. For all the numerical examples of this section, we set Ω to be a bounded open domain of $\mathbb{R}^2$. We let $\{\mathcal{T}_h\}_{h>0}$ be a family of quasi-uniform triangulations of Ω, which for a given meshsize $h > 0$, every element $K \in \mathcal{T}_h$ corresponds to a triangle of diameter $h_K \leq h$. For the spatial discretization, we make use of the continuous Lagrangian finite element space V_h of polynomial degree $\eta \geq 1$ introduced in (1.6) for both the wave and heat equations.

To handle the nonlinearity arising in the functional $\mathcal{N}$ present in the wave equation in (1.1), we used a fixed-point iteration method. In addition, to treat the nonlinear θ-dependent functions in the heat equation, we use a semi-implicit discretization in time. We let $\tau > 0$ be the timestep and define $t^n := n\tau$ as the discrete time points for all $n \in \mathbb{N}$, and use the superscript notation $(\cdot)^n$ to denote time evaluations with $t = t^n$. Regarding the time discretizations, given a time-dependent function $a = a(t)$, we consider the backward Euler scheme:

$$\partial_\tau a^{n+1} = \frac{1}{\tau}\big(a^{n+1} - a^n\big) \quad \text{and} \quad \partial_\tau^2 a^{n+1} = \frac{1}{\tau^2}\big(a^{n+1} - 2u^n + a^{n-1}\big), \tag{7.1}$$

and the second-order backward differentiation formulae (BDF2):

$$\begin{aligned} \partial_\tau a^{n+1} &= \frac{1}{\tau}\big(\tfrac{3}{2}a^{n+1} - 2a^n + \tfrac{1}{2}u^{n-1}\big), \\ \partial_\tau^2 a^{n+1} &= \frac{1}{\tau^2}\big(2a^{n+1} - 5a^n + 4a^{n-1} - a^{n-2}\big). \end{aligned} \tag{7.2}$$

We point out that in [1], it is shown that estimates as in Lemma 2.1 carry over to time discretization with the implicit Euler and the BDF2 scheme. To facilitate the writing of the fixed-point iteration method employed in the approximation of the wave equation, we define the following discrete operators

$$\delta_1 a^n := \begin{cases} a^n & \text{Euler}, \\ 2a^n - \frac{1}{2}a^{n-1} & \text{BDF2}, \end{cases} \qquad \delta_2 a^n := \begin{cases} 2a^n - a^{n-1} & \text{Euler}, \\ 5a^n - 4a^{n-1} + a^{n-2} & \text{BDF2}, \end{cases}$$

and the pair of constants $(\varsigma_1, \varsigma_2) = (1, 1)$ for the implicit Euler method and $(\varsigma_1, \varsigma_2) = (\frac{3}{2}, 2)$ for BDF2. Note that ς_1 and ς_2 are the numbers multiplying a^{n+1} in the first and second time derivatives, respectively, given for both time approximations. Then, with this notation, we can readily write

$$\partial_\tau a = \frac{1}{\tau}\big(\varsigma_1 a^{n+1} - \delta_1 a^n\big) \qquad \text{and} \qquad \partial_\tau^2 a = \frac{1}{\tau^2}\big(\varsigma_2 a^{n+1} - \delta_2 a^n\big). \tag{7.3}$$

Fully-discrete scheme. In order to perform the fixed-point iteration procedure, we first separate the terms containing the second-order time derivative of u in (1.3). For this purpose, we introduce two additional functionals $\mathcal{N}_1$ and $\mathcal{N}_2$, defined by

$$\mathcal{N}_1(\theta, u, u_t) = \begin{cases} 1 + 2k_{\mathrm{W}}(\theta)u & \text{Westervelt}, \\ 1 + 2k_{\mathrm{K}}(\theta)u_t & \text{Kuznetsov}, \end{cases} \qquad \mathcal{N}_2(\theta, u_t, \nabla u, \nabla u_t) = \begin{cases} 2k_{\mathrm{W}}(\theta)u_t^2 & \text{Westervelt}, \\ 2\nabla u \cdot \nabla u_t & \text{Kuznetsov}. \end{cases}$$

Observe that independent of the type of wave equation in (1.1), $\mathcal{N}_1$ corresponds to the nonlinear coefficient of $\partial_t^2 u$, while $\mathcal{N}_2$ contains the remaining non-linear terms of $\mathcal{N}$.

Next, given $u_h^n, u_h^{n-1}, \theta_h^n \in V_h$, and eventually $u_h^{n-2}, \theta_h^{n-1} \in V_h$ (cf. (7.2)), the fixed-point iteration procedure to update the wave equation from time t^n to t^{n+1}, is established as follows. We define auxiliary variables $u_h^{(i)} \in V_h$ with $i \in \mathbb{N}$, and set $u_h^{(0)} = u_h^n$. Then, given $u_h^{(i)}$, the iterative process continues by finding $u_h^{(i+1)} \in V_h$ such that

$$\begin{aligned}\big(\mathcal{N}_1^{(i)}\, \varsigma_2 u_h^{(i+1)}, \phi_h\big)_{L^2} + \tau^2 a\big(u_h^{(i+1)}, q(\theta_h^n)\phi_h\big) + \tau a\big(\varsigma_1 u_h^{(i+1)}, \beta(\theta_h^n)\phi_h\big) \\ = \big(\mathcal{N}_1^{(i)}\, \delta_2 u_h^n, \phi_h\big)_{L^2} + \tau a\big(\delta_1 u_h^n, \beta(\theta_h^n)\phi_h\big) - \tau^2\big(\mathcal{N}_2^{(i)}, \phi_h\big)_{L^2} + \tau^2 (f_h^{n+1}, \phi_h)_{L^2},\end{aligned} \tag{7.4}$$

for all $\phi_h \in V_h$, where the first and second time derivatives have been written using (7.3) to separate the variables arising in the iterative process from those of the discrete unknown at previous time steps, and

$$\begin{aligned}\mathcal{N}_1^{(i)} &= \mathcal{N}_1\Big(\theta_h^n, u_h^{(i)}, \tfrac{1}{\tau}\big(\varsigma_1 u_h^{(i)} - \delta_1 u_h^n\big)\Big), \\ \mathcal{N}_2^{(i)} &= \mathcal{N}_2\Big(\theta_h^n, \tfrac{1}{\tau}\big(\varsigma_1 u_h^{(i)} - \delta_1 u_h^n\big), \nabla u_h^{(i)}, \tfrac{1}{\tau}\nabla\big(\varsigma_1 u_h^{(i)} - \delta_1 u_h^n\big)\Big).\end{aligned}$$

The stopping criterion of the fixed-point iteration is set as follows:

$$\frac{\|u_h^{(i+1)} - u_h^{(i)}\|_{L^2(\Omega)}}{\|u_h^{(i+1)}\|_{L^2(\Omega)}} < \mathtt{tol},$$

and the unknown is updated by setting $u_h^{n+1} = u_h^{(i+1)}$. For the Pennes bioheat equation, we discretize the equation in a semi-implicit fashion, and we solve for $\theta_h^{n+1} \in V_h$ such that

$$\begin{aligned}(\varsigma_1 \theta_h^{n+1}, \phi_h)_{L^2} + \tau\kappa\, a(\theta_h^{n+1}, \phi_h) + \tau\nu(\theta_h^{n+1}, \phi_h)_{L^2} \\ = (\delta_1 \theta_h^n, \phi_h)_{L^2} + \tau\big(\mathcal{Q}(u_h^{n+1}, \partial_t u_h^{n+1}, \theta_h^n), \phi_h\big)_{L^2},\end{aligned} \tag{7.5}$$

for all $\phi_h \in V_h$. All numerical examples in this section have been implemented in Python using the open source finite element library FEniCSx [2]. For the fixed-point iteration method, we use the tolerance $\mathtt{tol} = 10^{-10}$. In addition, we remark that in order to properly initialize the BDF2 method, we perform the first time step with the implicit Euler method. The codes to reproduce the results are available at

`https://github.com/juliocareaga/wave-heat`

7.1. **Example 1: Accuracy tests.** To determine the numerical errors produced by our numerical scheme (7.4)–(7.5) along with time discretizations (7.1) or (7.2), we consider the unit square domain $\Omega = [0,1]^2$, and manufactured solutions $u_{\text{ex}} = u_{\text{ex}}(x,t)$ and $\theta_{\text{ex}} = \theta_{\text{ex}}(x,t)$. Then, the resulting terms arising after replacing $u = u_{\text{ex}}$ and $\theta = \theta_{\text{ex}}$ in (1.1) are supplemented to the wave and heat equations through the respective source terms:

$$\begin{aligned}f_{\text{ex}}(x,t) &= \partial_t^2 u_{\text{ex}} - c^2(\theta_{\text{ex}})\Delta u_{\text{ex}} - \beta(\theta_{\text{ex}})\Delta(\partial_t u_{\text{ex}}) \\ &\qquad + \mathcal{N}\big(u_{\text{ex}}, \partial_t u_{\text{ex}}, \partial_t^2 u_{\text{ex}}, \nabla u_{\text{ex}}, \nabla(\partial_t u_{\text{ex}}), \theta_{\text{ex}}\big), \\ g_{\text{ex}}(x,t) &= \partial_t \theta_{\text{ex}} - \kappa\Delta\theta_{\text{ex}} + \nu\theta_{\text{ex}} - \mathcal{Q}(u_{\text{ex}}, \partial_t u_{\text{ex}}, \theta_{\text{ex}}).\end{aligned}$$

We note that the forcing term g_{ex} was originally not present in the second equation of (1.1) and is introduced only for numerical testing. However, one could easily extend the error analysis above to this case, but we refrain from giving any details here. Given $\lambda_j, A_j > 0$ for $j \in \{1,2\}$, we define the following smooth manufactured solutions:

$$u_{\text{ex}}(x_1, x_2, t) = A_1 \sin(2\pi x_1)\sin(2\pi x_2)\exp(\lambda_1 t), \tag{7.6}$$

$$\theta_{\text{ex}}(x_1, x_2, t) = A_2 \sin(4\pi x_1)\sin(4\pi x_2)\exp(-\lambda_2 t), \tag{7.7}$$

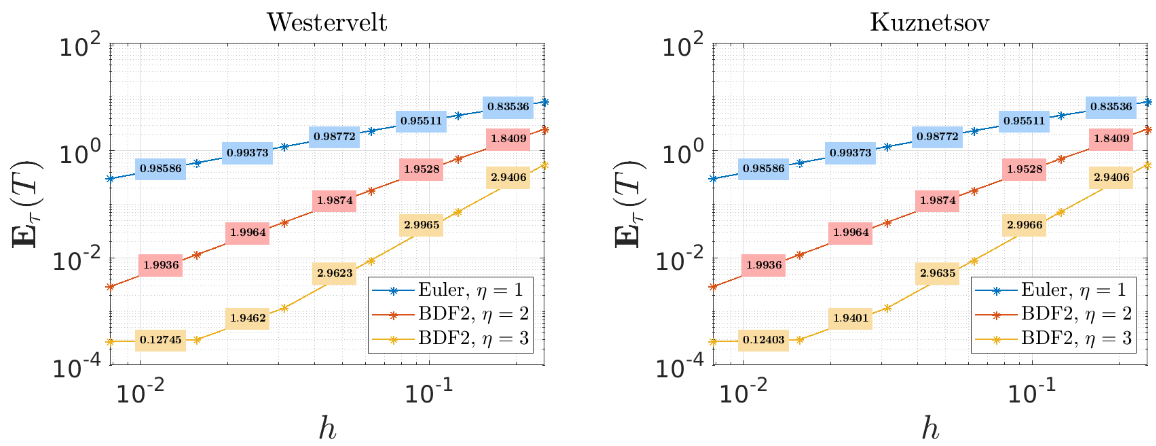

FIGURE 1. Total error (7.10) computed from the numerical scheme making use of the implicit Euler method with $\eta = 1$, and BDF2 method with $\eta = 2$, and $\eta = 3$, respectively. The errors are ploted against the meshsize h, for the common final time $T = 1\,\mathrm{s}$ and timestep $\tau = 1/128\,\mathrm{s} = 0.0078125\,\mathrm{s}$. The manufactured solutions are u_{ex} (cf. (7.6)) and θ_{ex} (cf. (7.7)) with $A_1 = 1$, $A_2 = 10^{-4}$, $\lambda_1 = 1$, and $\lambda_2 = 1/2$.

which satisfy the zero Dirichlet boundary conditions $u_{\mathrm{ex}|\partial\Omega} = \theta_{\mathrm{ex}|\partial\Omega} = 0$. In addition, we set the discrete initial conditions as

$$u_{0,h} = \mathrm{R}_h[u_{\mathrm{ex}}(x_1, x_2, 0)], \quad u_{1,h} = \mathrm{R}_h[\partial_t u_{\mathrm{ex}}(x_1, x_2, 0)], \quad \theta_{0,h} = \mathrm{R}_h[\theta_{\mathrm{ex}}(x_1, x_2, 0)],$$

for all $(x_1, x_2) \in \Omega$. For the error computations, we use a second order polynomial for the speed of sound (truncated polynomial function for liver tissue in [5]) and respective sound diffusivity function:

$$c(\theta) = 1529.3 + 1.6856\,(\theta + \Theta_{\mathrm{a}}) + 6.1131 \times 10^{-2}\,(\theta + \Theta_{\mathrm{a}})^2,$$
$$\beta(\theta) = \frac{2\tilde{\alpha}}{\omega^2} c^3(\theta),$$

with $\Theta_{\mathrm{a}} = 37\,°\mathrm{C}$ being the ambient temperature, $\tilde{\alpha} = 4.5 \times 10^{-6}\,\hat{f}\,\mathrm{Np\,m}^{-1}$ and $\omega = 2\pi\hat{f}$, for $\hat{f} = 1\,\mathrm{Hz}$. Furthermore, we set the coefficient functions k_{W} and k_{K} as

$$k_{\mathrm{W}}(\theta) = \frac{6}{\rho_{\mathrm{a}} c^2(\theta)}, \qquad k_{\mathrm{K}}(\theta) = \frac{5}{c^2(\theta)}, \tag{7.8}$$

where $B/(2A) = 5$ (cf. (1.2)). For the heat equation, we set the constants $\kappa = 1$ and $\nu = 10^{-5}$, the absorbed energy function

$$\mathcal{Q}(\theta, u, u_t) = \begin{cases} \dfrac{1}{2\rho_{\mathrm{a}}}\left(\dfrac{\tilde{\alpha}}{c(\theta)} u^2 + \dfrac{2b(\theta)}{c^4(\theta)} u_t^2\right) & \text{Westervelt,} \\ \dfrac{\rho_{\mathrm{a}}\tilde{\alpha}}{c(\theta)} u^2 & \text{Kuznetsov,} \end{cases} \tag{7.9}$$

with $\rho_{\mathrm{a}} = 1050\,\mathrm{kg/m}^3$. Given $\tau > 0$ and $t = t^{n+1}$, we define the total error associated with our coupled numerical scheme at the time $t = t^{n+1}$ as follows

$$\begin{aligned} \mathbf{E}_\tau(t^{n+1}) = {} & \left\|\nabla\left(\partial_t u(t^{n+1}) - \partial_\tau u_h^{n+1}\right)\right\|_{L^2(\Omega)} + \left\|\partial_t \theta(t^{n+1}) - \partial_\tau \theta_h^{n+1}\right\|_{L^2(\Omega)} \\ & + \left\|\nabla(\theta(t^{n+1}) - \theta_h^{n+1})\right\|_{L^2(\Omega)}, \end{aligned} \tag{7.10}$$

for all $n \geq 0$. Figure 1 shows the total error committed by our fully discrete numerical scheme for

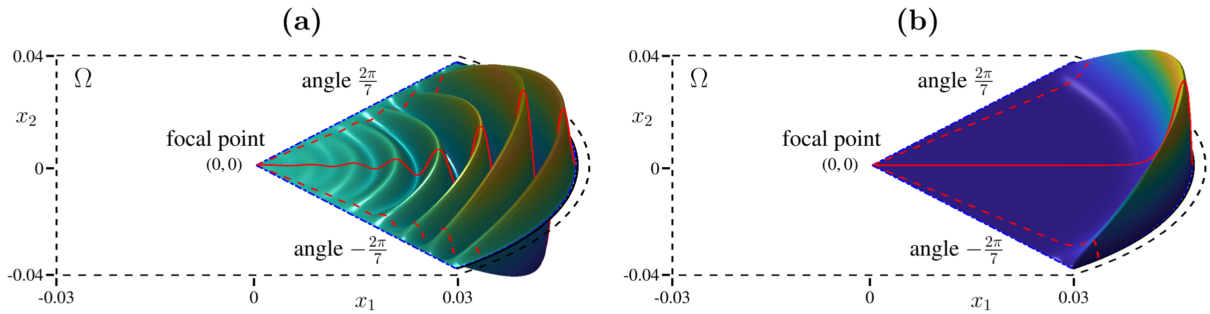

FIGURE 2. **(a)** Initial pressure $u_0 = u_0(x_1, x_2)$ used in Example 2, and **(b)** source function $f = f(x_1, x_2, t)$ at $t = 0$ employed in Example 3. The continuous red lines correspond to the respective functions at $x_2 = 0$ for $0 \leq x_1 \leq r_0$, and the red-dashed lines stand for the functions' plot at the angles $-\pi/4$ and $\pi/4$ respectively. The black-dashed contour coincide with the boundary $\partial\Omega$.

the two time approximations, implicit Euler and BDF2, with polynomial degrees $\eta = 1, 2, 3$, and for the Westervelt and Kuznetsov wave equations. For the two wave equations, the results show that in both cases, implicit Euler with $\eta = 1$, and BDF2 with $\eta = 2$, the orders of convergence are in agreement with what is predicted by Theorem 1.2. For the case of BDF2 with $\eta = 3$, we observe that the errors slopes tend to $\eta = 3$ until the order of convergence gets deteriorated due to the fact that τ overcomes h in the error $\mathcal{O}(\tau^2 + h^3)$. The later, explains the generation of a plateau as h reaches the smallest values.

7.2. **Example 2: Westervelt wave equation (initial excitation).** We let now Ω be composed by the union of the square region $[-0.03, 0.03]\times[-0.04, 0.04]$ and the circular segment between $-\pi/4$ and $\pi/4$ of the circle centered at the origin with radius 0.05 m. Therefore, Ω is not a polygonal domain and in this example (and in Example 3), we partially step aside from our theoretical framework. However, we still consider a polygonal approximation of the curved boundary of Ω to set the regular triangulation $\mathcal{T}_h$. The domain, shown in both plots of Figure 2 is delimited by the black-dashed lines accounting for $\partial\Omega$. For this example, we simulate the Westervelt equation with manufactured initial condition described by the functions (in polar coordinates)

$$\mathfrak{g}_0(r, \vartheta) = 10^6 \cos\left(\tfrac{7}{4}\vartheta\right) \tfrac{3\pi}{r_0} r \exp\left(-\tfrac{3\pi}{r_0} r\right) \sin\left(\tfrac{15\pi}{r_0} r\right),$$
$$\mathfrak{g}_1(r, \vartheta) = 10^6 \cos\left(\tfrac{7}{4}\vartheta\right) \tfrac{3\pi}{r_0} r \exp\left(-\tfrac{3\pi}{r_0} r\right) \cos\left(\tfrac{15\pi}{r_0} r\right),$$

where r is the radius measured from the origin and ϑ is the angle with respect to the x_1-axis, and $0.048\,\mathrm{m} = r_0 < 0.05\,\mathrm{m}$. Then, at $t = 0$, we set u and $\partial_t u$ as follows

$$(7.11) \qquad \big(u_0(r, \vartheta), u_1(r, \vartheta)\big) = \begin{cases} (\mathfrak{g}_0(r, \vartheta), \mathfrak{g}_1(r, \vartheta)) & \text{if } -\frac{2\pi}{7} \leq \vartheta \leq \frac{2\pi}{7}, \quad 0 \leq r \leq r_0, \\ (0, 0) & \text{otherwise in } \Omega, \end{cases}$$

and the zero initial temperature $\theta_0 \equiv 0$. Figure 2a displays the plot of u_0 as a function of (x_1, x_2) in Ω. We remark that the initial conditions are built in order to satisfy the zero Dirichlet boundary conditions and being continuous functions within the domain. The initial wave amplitude is maximal at the x_1-axis, when $\vartheta = 0$ (see the continuous red line in Figure 2) and it decreases to zero towards the lines $\vartheta = \pm 2\pi/7$ and $r = 0$.

For the coefficient functions in (1.1), we set the temperature-dependent speed of sound as the fifth order polynomial function modeling liver tissue in [5]

$$c(\theta) = 1529.3 + 1.6856\,(\theta + \Theta_a) + 6.1131 \times 10^{-2}\,(\theta + \Theta_a)^2$$
$$- 2.2967 \times 10^{-3}\,(\theta + \Theta_a)^3 + 2.2657 \times 10^{-5}\,(\theta + \Theta_a)^4 - 7.1795 \times 10^{-8}\,(\theta + \Theta_a)^5.$$

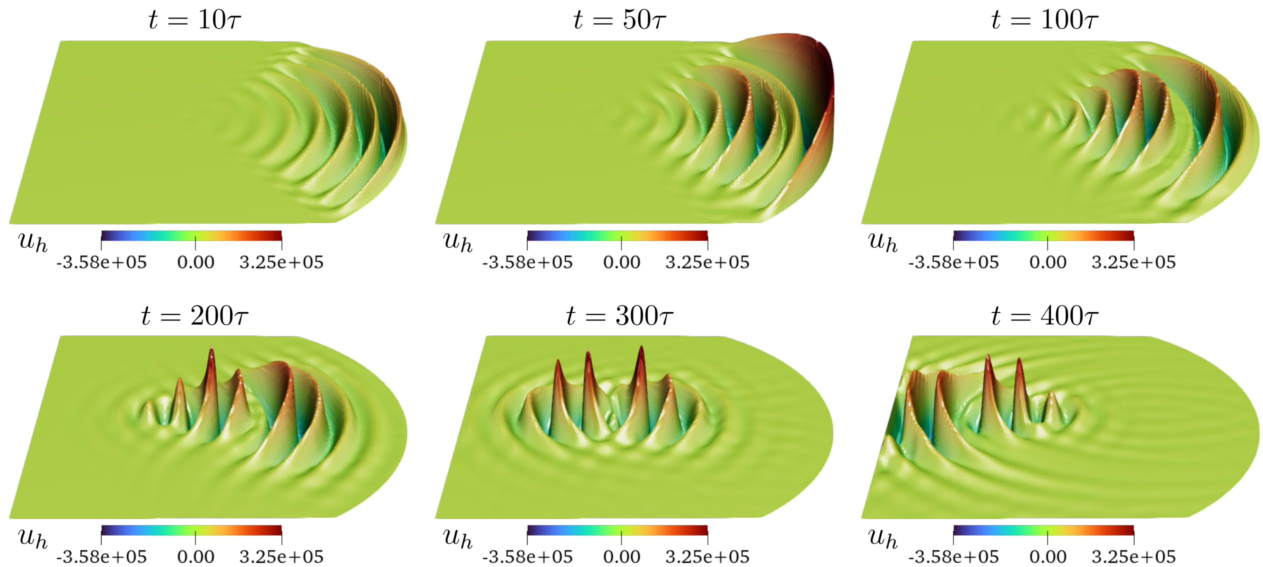

FIGURE 3. Example 2: Snapshots of the discrete pressure $u_h = u_h(x_1, x_2)$ computed with the Westervelt wave equation in (1.1), initial coditions (7.11) and $\theta_0 \equiv 0$, and setting the source term $f = 0$. The time step is $\tau = 10^{-7}\,$s and $\eta = 1$, and the time approximation is by the BDF2 method.

In addition, the corresponding sound diffusivity β and k_{W} function are taken as in Example 1 after setting the frequency $\hat{f} = 100\,$kHz. No source term is included in the wave equation in this example, i.e., $f \equiv 0$. The parameters used in the heat equation, which are related to liver tissue as the ambient 'a', and blood 'b' are taken from [5, Table 3] to be

$$\beta_{\mathrm{a}} = 6\,\mathrm{kg}^3\,\mathrm{m}^{-4}\,\mathrm{s}^{-2}, \quad \rho_{\mathrm{a}} = 1050\,\mathrm{kg}\,\mathrm{m}^{-3}, \quad \rho_{\mathrm{b}} = 1030\,\mathrm{kg}\,\mathrm{m}^{-3}, \quad \Theta_{\mathrm{a}} = 37\,^{\circ}\mathrm{C}$$
$$C_{\mathrm{a}} = 3600\,\mathrm{J}\,\mathrm{kg}\,\mathrm{K}^{-1}, \quad C_{\mathrm{b}} = 3620\,\mathrm{J}\,\mathrm{kg}\,\mathrm{K}^{-1}, \quad \kappa_{\mathrm{a}} = 0.512\,\mathrm{W}\,\mathrm{m}^{-1}\,\mathrm{K}^{-1},$$

and the constant diffusion κ and parameter ν are respectively given by

$$\kappa = \frac{\kappa_{\mathrm{a}}}{\rho_{\mathrm{a}} C_{\mathrm{a}}}, \qquad \nu = \frac{\rho_{\mathrm{b}} C_{\mathrm{b}}}{\rho_{\mathrm{a}} C_{\mathrm{a}}}.$$

For the absorbed acoustic energy functional, we use the definition given in (7.9).

For the numerical simulations, we use the BDF2 scheme with $\tau = 10^{-7}\,$s together with a polynomial approximation of degree $\eta = 1$, linear piecewise polynomials, and set $T = 4 \times 10^{-5}\,$s. The meshsize is set to $h = 7.6 \times 10^{-4}\,$m and the number of elements of the used mesh is 38912. In Figure 3, we show the simulated pressure profiles u_h^n at $n = 10, 50, 100, 200$ and 300. The combined effect of having the initial time derivative u_1, and u_0 given by (7.11) is that the wave amplitude reaches its maximum towards the focal point instead of directly dissipating, and travels towards the left boundary. Furthermore, it can be clearly seen that initially the acoustic wave traveling in the direction of the x_1-axis gets reflected from the curved boundary. The discrete temperature, on the other hand, is presented in Figure 4, in which it can be observed the heating effect that the ultrasound wave has on the focal area, where the temperature reaches its maximum.

7.3. **Example 3: Kuznetsov wave equation (source excitation).** In this example, we explore the case of a high frequency excitation due to a source term on the wave equation for the Kuznetsov equation. Unlike Example 2, the unknown u represents now the acoustic velocity potential. Under the same conditions of Example 2, meaning the same domain, parameters and coefficient functions

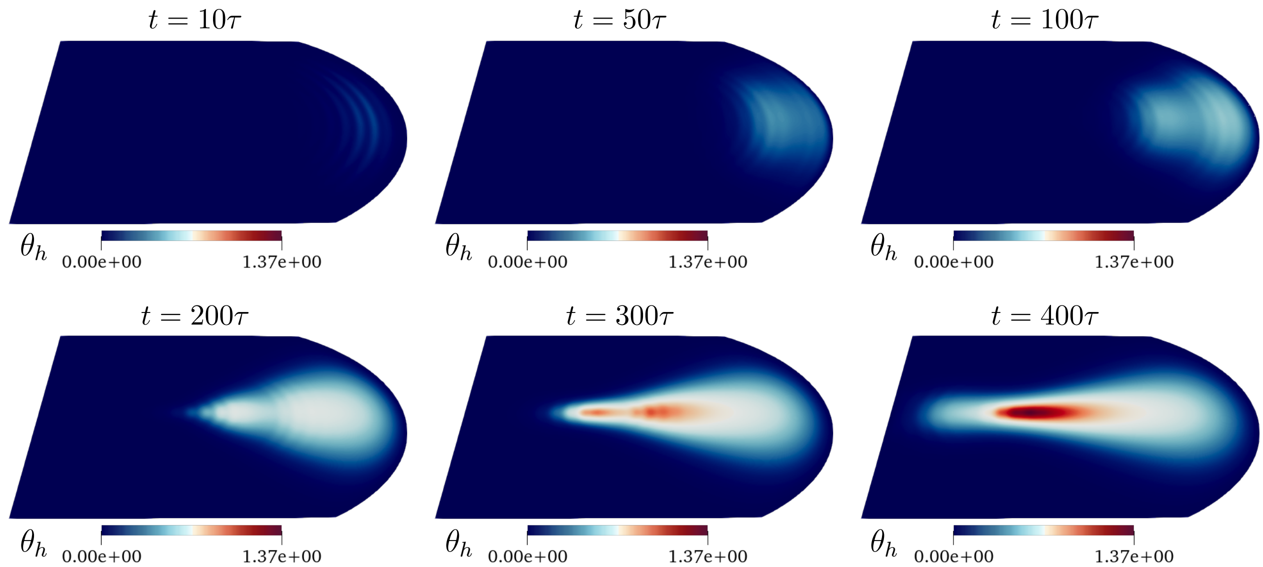

FIGURE 4. Example 2: Snapshots of the discrete temperature $\theta_h = \theta_h(x_1, x_2)$ computed with the Westervelt wave equation in (1.1), initial coditions (7.11) and $\theta_0 \equiv 0$, and setting the source term $f = 0$. The time step is $\tau = 10^{-7}\,$s and $\eta = 1$, and the time approximation is by the BDF2 method.

(with k_{K} as in (7.8)), we set the source term function similarly as the initial conditions in Example 2 through the function (in polar coordinates)

$$\mathfrak{f}_0(r, \vartheta, t) = 10^8 \cos\left(\tfrac{7}{4}\vartheta\right)\tfrac{1}{r_0} r \Big(\exp\left(-40r/r_0\right) - \exp\left(-40\right)\Big)\cos(\omega t), \tag{7.12}$$

where the pair (r, ϑ) corresponds to the polar coordinates relative to $(x_1, x_2) \in \Omega$, and $r_0 = 0.048\,$m. Then, we define the source term function as follows:

$$f(r, \vartheta, t) = \begin{cases} \mathfrak{f}_0(r, \vartheta, t) & \text{if } -\frac{2\pi}{7} \leq \vartheta \leq \frac{2\pi}{7}, \quad 0 \leq r \leq r_0, \\ 0 & \text{otherwise in } \Omega. \end{cases}$$

Figure 2b) shows the plot of function f at the time $t = 0$. Varying time t, the described source term oscillates with angular frequency $\omega = 2\pi\hat{f}$, with $\hat{f} = 100\,$kHz. This function is intended to mimic the effect of having an excitation due to Neumann boundary conditions, but keeping the unknowns to zero at the boundaries. To perform this numerical example, we use the BDF2 method with $\tau = 10^{-7}\,$s, $T = 4 \times 10^{-5}\,$s, linear elements with $\eta = 1$, and the same mesh as in Example 1.

In Figure 5, we show snapshots of the discrete solution u_h (top row), and discrete temperature (bottom row) at three times t^n with $n = 200, 300$ and 400. Unlike the previous example, in which the wave is induced by the initial condition, in this case the oscillations of the source term continuously drive the wave. Then, sequential peaks of u_h reach the focal area as time evolves. The approximated temperature θ_h is presented in Figure 5, which is in the order of magnitude of $10^{-9}\,$°C. The later can be explained due to the reduced magnitude of u_h, which directly influences the strength of the absorbed energy function $\mathcal{Q}$.

ACKNOWLEDGMENTS

Funding. J.C. is supported by ANID through Fondecyt project 3230553. B.D. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 258734477 – SFB 1173.

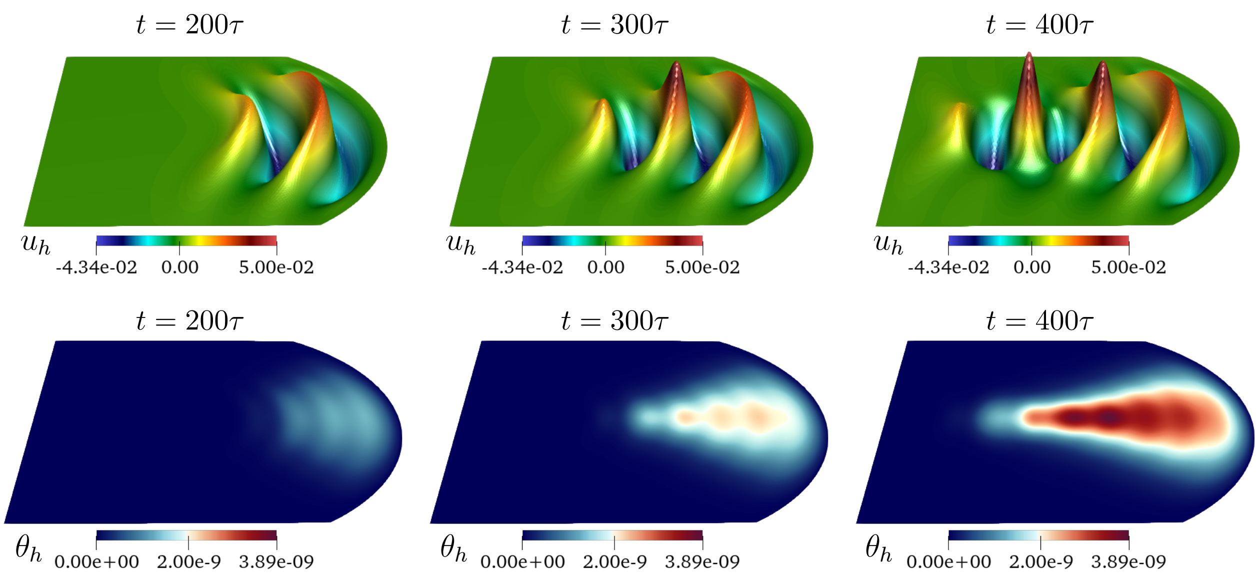

FIGURE 5. Example 2: Snapshots of the discrete acoustic velocity potential $u_h = u_h(x_1, x_2)$ (first row) and discrete temperature $\theta_h = \theta_h(x_1, x_2)$ (second row), at three time points, computed with the Kuznetsov wave equation in (1.1) with zero initial conditions and source term f given by (7.12). The time step is $\tau = 10^{-7}$ s and $\eta = 1$, and the time approximation is by the BDF2 method.

Appendix A. Postponed proofs

In this section, we finally give the proofs from Sections 2.4 and 5 on the bound of the Ritz projection and the defects from the error analysis of the heat part.

Proof of Lemma 2.3. We add and subtract $\mu(\psi_h)\,\phi_h$ to obtain

$$\|\mathrm{R}_h[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)} \leq \|\mu(\psi_h)\,\phi_h\|_{L^2(\Omega)} + \|(\mathrm{I}-\mathrm{R}_h)[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)},$$

which yields

$$\|\mathrm{R}_h[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)} \leq \|\mu(\psi_h)\|_{L^\infty(\Omega)}\|\phi_h\|_{L^2(\Omega)} + \|(\mathrm{I}-\mathrm{R}_h)[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)}.$$

To estimate the second term on the right-hand side, we note that

$$\|(\mathrm{I}-\mathrm{R}_h)[\mu(\psi_h)\,\phi_h]\|_{L^2(\Omega)} \lesssim h\|(\mathrm{I}-\mathrm{R}_h)[\mu(\psi_h)\,\phi_h]\|_{H^1(\Omega)} \lesssim h\|(\mathrm{I}-\mathrm{I}_h)[\mu(\psi_h)\,\phi_h]\|_{H^1(\Omega)}.$$

Using the standard interpolation estimate, we obtain

$$h^2\|(\mathrm{I}-\mathrm{I}_h)(\mu(\psi_h)\,\phi_h)\|^2_{H^1(\Omega)} \leq C\sum_K \big(h^{\eta+1}|\mu(\psi_h)\,\phi_h|_{H^{\eta+1}(K)}\big)^2.$$

We use that on each cell ϕ_h is a polynomial of degree η, together with the inverse estimate in [4, Lemma (4.5.3)], to derive

$$h^{\eta+1}|\mu(\psi_h)\,\phi_h|_{H^{\eta+1}(K)} \lesssim h^{\eta+1}\sum_{j=1}^{\eta+1}|\mu(\psi_h)|_{W^{j,\infty}(K)}|\phi_h|_{H^{\eta+1-j}(K)} \lesssim |\phi_h|_{L^2(K)}\sum_{j=1}^{\eta+1}h^j|\mu(\psi_h)|_{W^{j,\infty}(K)}.$$

The observation that

$$h^j|\mu(\psi_h)|_{W^{j,\infty}(K)} \lesssim \sum_{\ell=1}^{j} h^\ell\|\nabla\psi_h\|^\ell_{L^\infty(K)}$$

allows us to conclude

$$h^2\|(\mathrm{I}-\mathrm{I}_h)(\mu(\psi_h)\,\phi_h)\|^2_{H^1(\Omega)} \leq C\sum_K \|\phi_h\|^2_{L^2(K)}\sum_{j=1}^{\eta+1} h^{2j}\|\nabla\psi_h\|^{2j}_{L^\infty(K)} \leq C\|\phi_h\|^2_{L^2(\Omega)}\sum_{j=1}^{\eta+1}\big(h\|\nabla\psi_h\|_{L^\infty(\Omega)}\big)^{2j}.$$

Finally, we employ

$$h\|\nabla\psi_h\|_{L^\infty(\Omega)} \lesssim h^{\delta/(d+\delta)}\|\nabla\psi_h\|_{W^{1,d+\delta}(\Omega)}$$

to obtain the claim. □

Next, we turn to the defect defined in (5.1).

Proof of Lemma 5.1. (a) We have the rewriting

$$\begin{aligned}(\delta^\theta,\phi_h)_{L^2} =&\, (\partial_t\mathrm{R}_h\theta-\theta_t,\phi_h)_{L^2}+\nu(\mathrm{R}_h\theta-\theta,\phi_h)_{L^2}+((\alpha(\theta)-\alpha(\mathrm{R}_h\theta))(\zeta_1u^2+\zeta_2u_t^2),\phi_h)_{L^2}\\ &+(\alpha(\mathrm{R}_h\theta)(\zeta_1(u-\mathrm{R}_hu)(u+\mathrm{R}_hu)+\zeta_2(u_t-\partial_t\mathrm{R}_hu)(u_t+\partial_t\mathrm{R}_hu)),\phi_h)_{L^2}.\end{aligned}$$

The statement then follows by the approximation properties of the Ritz projection.

(b) For the time derivative of the defect, we note that

$$\begin{aligned}(\partial_t\delta^\theta,\phi_h)_{L^2} =&\, (\partial_t^2\mathrm{R}_h\theta-\theta_{tt},\phi_h)_{L^2}+\nu(\partial_t\mathrm{R}_h\theta-\theta_t,\phi_h)_{L^2}\\ &+(\alpha'(\theta)\theta_t(\zeta_1u^2+\zeta_2u_t^2)-\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2),\phi_h)_{L^2}\\ &-2(\alpha(\theta)u_t(\zeta_1u+\zeta_2u_{tt})-\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_hu(\zeta_1\mathrm{R}_hu+\zeta_2\partial_t^2\mathrm{R}_hu),\phi_h)_{L^2}.\end{aligned}$$

We have the following rewriting:

$$\begin{aligned}&(\alpha'(\theta)\theta_t(\zeta_1u^2+\zeta_2u_t^2)-\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2),\phi_h)_{L^2}\\ &=((\alpha'(\theta)-\alpha'(\mathrm{R}_h\theta))\theta_t(\zeta_1u^2+\zeta_2u_t^2),\phi_h)_{L^2}\\ &\quad+(\alpha'(\mathrm{R}_h\theta)(\theta_t-\partial_t\mathrm{R}_h\theta)(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2),\phi_h)_{L^2}\\ &\quad+(\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(u_h-\mathrm{R}_hu)(u_h+\mathrm{R}_hu)+\zeta_2(\partial_tu_h-\partial_t\mathrm{R}_hu)(\partial_tu_h+\partial_t\mathrm{R}_hu)),\phi_h)_{L^2}.\end{aligned}\tag{A.1}$$

On account of the local Lipschitz continuity of α', we have

$$\begin{aligned}&\|(\alpha'(\theta)-\alpha'(\mathrm{R}_h\theta))\theta_t(\zeta_1u^2+\zeta_2u_t^2)\|_{L^2(L^2(\Omega))}\\ &\lesssim\|\alpha'(\theta)-\alpha'(\mathrm{R}_h\theta)\|_{L^\infty(L^2(\Omega))}\|\theta_t\|_{L^2(L^\infty(\Omega))}(\|u\|^2_{L^\infty(L^\infty(\Omega))}+\|u_t\|^2_{L^\infty(L^\infty(\Omega))})\\ &\lesssim\|\theta-\mathrm{R}_h\theta\|_{L^\infty(L^2(\Omega))}\|\theta_t\|_{L^2(L^\infty(\Omega))}(\|u\|^2_{L^\infty(L^\infty(\Omega))}+\|u_t\|^2_{L^\infty(L^\infty(\Omega))})\end{aligned}$$

and we can estimate the other terms on the right-hand side of (A.1) in an analogous manner. Similarly, we have the rewriting

$$\begin{aligned}&(\alpha(\theta)u_t(\zeta_1u+\zeta_2u_{tt})-\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_hu(\zeta_1\mathrm{R}_hu+\zeta_2\partial_t^2\mathrm{R}_hu),\phi_h)_{L^2}\\ &=((\alpha(\theta)-\alpha(\mathrm{R}_h\theta))u_t(\zeta_1u+\zeta_2u_{tt}),\phi_h)_{L^2}+(\alpha(\mathrm{R}_h\theta)(u_t-\partial_t\mathrm{R}_hu)(\zeta_1u+\zeta_2u_{tt}),\phi_h)_{L^2}\\ &\quad+(\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_hu(\zeta_1(u-\mathrm{R}_hu)+\zeta_2(u_{tt}-\partial_t^2\mathrm{R}_hu)),\phi_h)_{L^2}\end{aligned}$$

and we can proceed as above to arrive at the claim. □

The last estimate deals with the right-hand side in the error equation of the heat problem.

Proof of Lemma 5.2. We use the following rewriting:

$$\begin{aligned}(\mathcal{F}_h^{\theta_t},\phi_h)_{L^2} =&\, (\delta^\theta,\phi_h)_{L^2}+((\alpha(\mathrm{R}_h\theta)-\alpha(\theta_h))(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)),\phi_h)_{L^2}\\ &+(\alpha(\theta_h)(\zeta_1(\mathrm{R}_hu-u_h)(\mathrm{R}_hu+u_h)+\zeta_2(\partial_t\mathrm{R}_hu-\partial_tu_h)(\partial_t\mathrm{R}_hu+\partial_tu_h)),\phi_h)_{L^2}.\end{aligned}$$

We further have

$$\begin{aligned}&\|(\alpha(\mathrm{R}_h\theta)-\alpha(\theta_h))(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2))\|_{L^2_t(L^2(\Omega))}\\ &\lesssim\|e_h^\theta\|_{L^2_t(L^2(\Omega))}\|\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2\|_{L^\infty_t(L^\infty(\Omega))},\end{aligned}$$

where we have relied on

$$\|\alpha(\mathrm{R}_h\theta)-\alpha(\theta_h)\|_{L^2_t(L^2(\Omega))}\lesssim\|\mathrm{R}_h\theta-\theta_h\|_{L^2_t(L^2(\Omega))}$$

for $\|\theta_h\|_{L^\infty_t(L^\infty(\Omega))}\lesssim 1$. Similarly,

$$\begin{aligned}&\|\alpha(\theta_h)(\zeta_1(\mathrm{R}_hu-u_h)(\mathrm{R}_hu+u_h)+\zeta_2(\partial_t\mathrm{R}_hu-\partial_tu_h)(\partial_t\mathrm{R}_hu+\partial_tu_h))\|_{L^2_t(L^2(\Omega))}\\ \lesssim\,&\|\alpha(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}(\|e^u_h\|_{L^2_t(L^2(\Omega))}+\|\partial_te^u_h\|_{L^2_t(L^2(\Omega))})(\|u_h\|_{H^1_t(L^2(\Omega))}+\|\mathrm{R}_hu\|_{H^1_t(L^2(\Omega))}).\end{aligned}$$

The claim then follows by Lemma 5.1 and the properties of the Ritz projection.

We now tackle the estimate of $\partial_t\mathcal{F}^\theta_h$. Note that

$$(\partial_t\mathcal{F}^\theta_h,\phi_h)_{L^2}=\,(\partial_t\delta^\theta,\phi_h)_{L^2}+\mathcal{I},$$

where

$$\begin{aligned}\mathcal{I}=\,&(\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)+2\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_hu(\zeta_1\mathrm{R}_hu+\zeta_2\partial_t^2\mathrm{R}_hu),\phi_h)_{L^2}\\&-(\alpha'(\theta_h)\partial_t\theta_h(\zeta_1u_h^2+\zeta_2\partial_tu_h^2)+2\alpha(\theta_h)\partial_tu_h(\zeta_1u_h+\zeta_2\partial_t^2u_h),\phi_h)_{L^2}.\end{aligned}$$

We can rewrite $\mathcal{I}$ as follows:

$$\begin{aligned}\mathcal{I}=\,&(\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)-\alpha'(\theta_h)\partial_t\theta_h(\zeta_1u_h^2+\zeta_2\partial_tu_h^2),\phi_h)_{L^2}\\&+2(\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_hu(\zeta_1\mathrm{R}_hu+\zeta_2\partial_t^2\mathrm{R}_hu)-\alpha(\theta_h)\partial_tu_h(\zeta_1u_h+\zeta_2\partial_t^2u_h),\phi_h)_{L^2}.\end{aligned}$$

We next further rewrite the two difference terms. First,

$$\begin{aligned}&(\alpha'(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)-\alpha'(\theta_h)\partial_t\theta_h(\zeta_1u_h^2+\zeta_2\partial_tu_h^2),\phi_h)_{L^2}\\=\,&((\alpha'(\mathrm{R}_h\theta)-\alpha'(\theta_h))\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2),\phi_h)_{L^2}\\&+(\alpha'(\theta_h)(\partial_t\mathrm{R}_h\theta-\partial_t\theta_h)(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2),\phi_h)_{L^2}\\&+\zeta_1(\alpha'(\theta_h)\partial_t\theta_h(\mathrm{R}_hu-u_h)(\mathrm{R}_hu+u_h),\phi_h)_{L^2}\\&+\zeta_2(\alpha'(\theta_h)\partial_t\theta_h(\partial_t\mathrm{R}_hu-\partial_tu_h)(\partial_t\mathrm{R}_hu+\partial_tu_h)),\phi_h)_{L^2}:=\sum_{i=1}^4(\mathcal{I}^i,\phi_h)_{L^2}.\end{aligned}$$

We have

$$\begin{aligned}&\|\mathcal{I}^1\|_{L^2_t(L^2(\Omega))}\\&=\|(\alpha'(\mathrm{R}_h\theta)-\alpha'(\theta_h))\partial_t\mathrm{R}_h\theta(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)\|_{L^2_t(L^2(\Omega))}\\&\lesssim\|\alpha'(\mathrm{R}_h\theta)-\alpha'(\theta_h)\|_{L^2_t(L^2(\Omega))}\|\partial_t\mathrm{R}_h\theta\|_{L^\infty(L^\infty(\Omega))}(\|\mathrm{R}_hu\|^2_{L^\infty(L^\infty(\Omega))}+\|\partial_t\mathrm{R}_hu\|^2_{L^\infty(L^\infty(\Omega))})\\&\lesssim\|\theta\|_{\mathcal{X}_\theta}\|u\|^2_{\mathcal{X}_u}\|e^\theta_h\|_{L^2_t(L^2(\Omega))}.\end{aligned}$$

Secondly,

$$\begin{aligned}\|\mathcal{I}^2\|_{L^2_t(L^2(\Omega))}&=\|\alpha'(\theta_h)(\partial_t\mathrm{R}_h\theta-\partial_t\theta_h)(\zeta_1(\mathrm{R}_hu)^2+\zeta_2(\partial_t\mathrm{R}_hu)^2)\|_{L^2_t(L^2(\Omega))}\\&\lesssim\|\alpha'(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}\|\partial_te^\theta_h\|_{L^2_t(L^2(\Omega))}(\|\mathrm{R}_hu\|^2_{L^\infty(L^\infty(\Omega))}+\|\partial_t\mathrm{R}_hu\|^2_{L^\infty(L^\infty(\Omega))}).\end{aligned}$$

Thirdly,

$$\begin{aligned}\|\mathcal{I}^3\|_{L^2_t(L^2(\Omega))}&=\|\zeta_1\alpha'(\theta_h)\partial_t\theta_h(\mathrm{R}_hu-u_h)(\mathrm{R}_hu+u_h)\|_{L^2_t(L^2(\Omega))}\\&\lesssim\|\partial_t\theta_h\|_{L^\infty_t(L^3(\Omega))}\|e^u_h\|_{L^2_t(L^6(\Omega))}\lesssim\|\partial_t\theta_h\|_{L^\infty_t(L^3(\Omega))}\|\nabla e^u_h\|_{L^2_t(L^2(\Omega))}.\end{aligned}$$

Next,

$$\begin{aligned}\|\mathcal{I}^4\|_{L^2_t(L^2(\Omega))}&=\|\zeta_2\alpha'(\theta_h)\partial_t\theta_h(\partial_t\mathrm{R}_hu-\partial_tu_h)(\partial_t\mathrm{R}_hu+\partial_tu_h)\|_{L^2_t(L^2(\Omega))}\\&\lesssim\|\partial_t\theta_h\|_{L^\infty_t(L^3(\Omega))}\|\partial_te^u_h\|_{L^2_t(L^6(\Omega))}\lesssim\|\partial_t\theta_h\|_{L^\infty_t(L^3(\Omega))}\|\nabla\partial_te^u_h\|_{L^2_t(L^2(\Omega))}.\end{aligned}$$

Similarly, we have the following rewriting:

$$\begin{aligned}
&2(\alpha(\mathrm{R}_h\theta)\partial_t\mathrm{R}_h u(\zeta_1\mathrm{R}_h u+\zeta_2\partial_t^2\mathrm{R}_h u)-\alpha(\theta_h)\partial_t u_h(\zeta_1 u_h+\zeta_2\partial_t^2 u_h),\phi_h)_{L^2}\\
&=2((\alpha(\mathrm{R}_h\theta)-\alpha(\theta_h))\partial_t\mathrm{R}_h u(\zeta_1\mathrm{R}_h u+\zeta_2\partial_t^2\mathrm{R}_h u),\phi_h)_{L^2}\\
&\quad+2(\alpha(\theta_h)(\partial_t\mathrm{R}_h u-\partial_t u_h)(\zeta_1\mathrm{R}_h u+\zeta_2\partial_t^2\mathrm{R}_h u),\phi_h)_{L^2}\\
&\quad+2(\alpha(\theta_h)\partial_t u_h(\zeta_1(\mathrm{R}_h u-u_h)+\zeta_2(\partial_t^2\mathrm{R}_h u-\partial_t^2 u_h)),\phi_h)_{L^2}\\
&:=\sum_{i=5}^{7}(\mathcal{I}^i,\phi_h)_{L^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{I}^5\|_{L^2_t(L^2(\Omega))}&=\|2(\alpha(\mathrm{R}_h\theta)-\alpha(\theta_h))\partial_t\mathrm{R}_h u(\zeta_1\mathrm{R}_h u+\zeta_2\partial_t^2\mathrm{R}_h u)\|_{L^2_t(L^2(\Omega))}\\
&\lesssim\|e_h^\theta\|_{L^2_t(L^\infty(\Omega))}\|\partial_t\mathrm{R}_h u\|_{L^\infty_t(L^\infty(\Omega))}(\|\mathrm{R}_h u\|_{L^2(L^2(\Omega))}+\|\partial_t^2\mathrm{R}_h u\|_{L^2(L^2(\Omega))}).
\end{aligned}$$

Next,

$$\begin{aligned}
\|\mathcal{I}^6\|_{L^2_t(L^2(\Omega))}&=\|2\alpha(\theta_h)(\partial_t\mathrm{R}_h u-\partial_t u_h)(\zeta_1\mathrm{R}_h u+\zeta_2\partial_t^2\mathrm{R}_h u)\|_{L^2_t(L^2(\Omega))}\\
&\lesssim\|\alpha(\theta_h)\|_{L^\infty_t(L^\infty(\Omega))}\|\partial_t e_h^\theta\|_{L^\infty_t(L^6(\Omega))}(\|\mathrm{R}_h u\|_{L^2(L^3(\Omega))}+\|\partial_t^2\mathrm{R}_h u\|_{L^2(L^3(\Omega))}).
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\mathcal{I}^7\|_{L^2_t(L^2(\Omega))}&=\|2\alpha(\theta_h)\partial_t u_h(\zeta_1(\mathrm{R}_h u-u_h)+\zeta_2(\partial_t^2\mathrm{R}_h u-\partial_t^2 u_h))\|_{L^2_t(L^2(\Omega))}\\
&\lesssim\|\alpha(\theta_h)\,\partial_t u_h\|_{L^\infty_t(L^\infty(\Omega))}(\|e_h^u\|_{L^2_t(L^2(\Omega))}+\|\partial_t^2 e_h^u\|_{L^2_t(L^2(\Omega))}).
\end{aligned}$$

Combining the derived bounds yields the desired result. □

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